A CATEGORICAL INVARIANT FOR CUBIC THREEFOLDS

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Abstract. We prove a categorical version of the Torelli theorem for cubic threefolds. More precisely, we show that the non-trivial part of a semi-orthogonal decomposition of the derived category of a cubic threefold characterizes its isomorphism class.

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1. Introduction

One of the main ideas in derived category theory, which goes back to Bondal and Orlov, is that the bounded derived category $\mathrm{D}^{\mathrm{b}}(X)$ of coherent sheaves on a smooth projective variety X should contain interesting information about the geometry of the variety itself, for example, about its birational properties.

The main problem is that such information is encoded in a rather inexplicit way. A general belief is that part of this information could be obtained by looking at semi-orthogonal decompositions (see Definition 2.1)

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathbf{T}_1, \ldots, \mathbf{T}_n \rangle,$$

where the $\mathbf{T}_i$'s are full triangulated subcategories of $\mathrm{D}^{\mathrm{b}}(X)$ satisfying some orthogonality conditions. In many interesting geometric situations, all the categories $\mathbf{T}_i$ but one are equivalent to the derived category of a point. In the easiest case of projective spaces, one can obtain decompositions in which all the subcategories $\mathbf{T}_i$ are of this form. In general, a non-trivial subcategory is present and carries useful information about X. This happens, for example, when X is the intersection of two quadrics of even dimension [2]. In that case, the non-trivial subcategory is equivalent to $\mathrm{D}^{\mathrm{b}}(C)$, where C is the curve which is the fine moduli space of spinor bundles on the pencil generated by the two quadrics.

The same strategy has been pursued by Kuznetsov in a series of papers to study the derived categories of Fano threefolds. Another interesting example is the one of X a V_{14} Fano threefold, i.e. a smooth complete intersection of $\mathbb{P}^9$ and the Grassmannian $\mathrm{Gr}(2,6)$ in $\mathbb{P}^{14}$. These are the Fano threefolds with $\mathrm{Pic}(X) = \mathbb{Z}$, index 1, and genus 8. A classical construction shows that there is a correspondence between birational classes of V_{14} Fano threefolds and isomorphism classes of cubic threefolds (i.e. smooth hypersurfaces of degree 3 in $\mathbb{P}^4$). Let X be a Fano threefold as above and let Y be the cubic threefold related to X by the previous correspondence. By [15], we have a semi-orthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(Y)$ as

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle,$$

where $\mathcal{O}_Y(1) := \mathcal{O}_{\mathbb{P}^4}(1)|_Y$ and $\mathbf{T}_Y := \langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle^{\perp}$. Moreover, $\mathbf{T}_Y$ is an example of a Calabi-Yau category of dimension 5/3: the composition of three copies of the Serre functor $S_{\mathbf{T}_Y}$ is isomorphic to the shift by 5. At the same time, again by [15], the category $\mathrm{D}^{\mathrm{b}}(X)$ has a semi-orthogonal decomposition too whose unique non-trivial part $\mathbf{T}_X$ is equivalent to $\mathbf{T}_Y$. As a consequence, Kuznetsov deduces that $\mathbf{T}_X$ is a birational invariant for V_{14} Fano threefolds. A natural question is now whether $\mathbf{T}_X$ is a "good" invariant which characterizes the birational class of X.

By the correspondence mentioned above, one can then forget about X and just study the cubic hypersurface Y. In particular, the problem of $\mathbf{T}_X$ being a good invariant translates into the problem

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of $\mathbf{T}_Y$ being a good isomorphism invariant of Y. The main result of this paper (conjectured by Kuznetsov in [15]) gives a complete answer to this question and can be regarded as the categorical version of the classical Clemens-Griffiths-Tyurin Torelli Theorem for cubic threefolds:

Theorem 1.1. *Two cubic threefolds Y_1 and Y_2 are isomorphic if and only if $\mathbf{T}_{Y_1}$ and $\mathbf{T}_{Y_2}$ are equivalent triangulated categories.*

Since the Picard group of a cubic threefold is free of rank one, then $Y_1 \cong Y_2$ clearly implies that $\mathbf{T}_{Y_1} \cong \mathbf{T}_{Y_2}$. To prove the only non-trivial implication of Theorem 1.1, we will follow an idea of Kuznetsov [15], which can be roughly summarized as follows: The ideal sheaves I_l of lines l in Y are all inside the category $\mathbf{T}_Y$. We will define a stability condition on $\mathbf{T}_Y$ in such a way that any stable object in $\mathbf{T}_Y$ numerically equivalent to I_l is actually isomorphic to some $I_{l'}$. In such a way, the Fano surface of lines $F(Y)$ of Y is realized as a moduli space of stable objects in $\mathbf{T}_Y$ and it is possible to reconstruct the intermediate Jacobian $J(Y)$ of Y from $\mathbf{T}_Y$ (being $J(Y)$ the Albanese variety of $F(Y)$). Theorem 1.1 will then follow from the Torelli Theorem for cubic threefold [7].

Theorem 1.1 can be also interpreted as a supporting evidence for a conjecture of Kuznetsov about the rationality of cubic fourfolds (i.e. smooth hypersurfaces of degree 3 in $\mathbb{P}^5$). If Y is such a fourfold, the results of [15] yield another semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle,$$

where $\mathcal{O}_Y(i)$ and $\mathbf{T}_Y$ are defined as above. Notice that $\mathbf{T}_Y$ is a Calabi-Yau category of dimension 2. Kuznetsov conjectures in [17] that this category may encode a fundamental geometric property of the fourfold:

Conjecture 1.2. (Kuznetsov) *A cubic fourfold Y is rational if and only if $\mathbf{T}_Y$ is equivalent to the derived category of a K3 surface.*

This conjecture can be regarded as a categorical counterpart in higher dimension of the construction of the intermediate Jacobian for a threefold. Indeed, according to [7], the intermediate Jacobian becomes a birational invariant once one forgets all its irreducible factors coming from curves (and points). The category $\mathbf{T}_Y$ should have precisely the same flavor: after discarding the part coming from points, curves and surfaces, what remains is expected to be a birational invariant [17]. Indeed, in the rational case, the category $\mathbf{T}_Y$ is not a birational invariant (see, for example, Proposition 5.5), while the expected birational invariant is trivial.

In the threefold case, Theorem 1.1 fits this picture: it can be interpreted by saying that the category $\mathbf{T}_Y$ carries precisely the same information as the intermediate Jacobian of Y.

The main evidence for Conjecture 1.2 is that, in [17], it has been verified for all the basic examples of rational cubic fourfolds: the Pfaffian cubics, the cubic fourfolds containing a plane described in [9] and the singular cubics. If the previous conjecture is true, then it would follow from the calculations in the appendix of [17] that, generically, cubic fourfolds are not rational. In Section 5.3 we will clarify how an analogue of Theorem 1.1 can be stated for cubic fourfolds containing a plane.

The plan of the paper is as follows. Sections 2 and 3 are mainly devoted to the construction of a bounded t-structure on $\mathbf{T}_Y$. This is the first step toward the construction of a suitable stability condition on this category. Many ingredients will play a role in these two sections, among which Kuznetsov's results about quadric fibrations, the description of the numerical Grothendieck group of $\mathbf{T}_Y$ and the definition of a slope-stability for sheaves on the projective plane which are also modules over a certain algebra.

Section 4 deals with the construction of a stability condition on $\mathbf{T}_Y$. The definition is quite natural but what requires much more work is to show that the ideal sheaves of lines in Y (which are all objects of $\mathbf{T}_Y$) are all stable in this stability condition and that they are (up to even shifts)

the only stable objects in their numerical class. In Section 5 we then conclude the proof of Theorem 1.1 and examine the case of cubic fourfolds containing a plane.

Notation. In this paper all varieties are defined over the complex numbers $\mathbb{C}$ and all triangulated categories are assumed to be essentially small (i.e. equivalent to a small category) and linear over $\mathbb{C}$ (i.e. all Hom spaces are $\mathbb{C}$-vector spaces). For a variety X, $\mathrm{D^b}(X) := \mathrm{D^b}(\mathbf{Coh}(X))$ is the bounded derived category of coherent sheaves on X. All derived functors will be denoted as if they were underived, e.g. for a morphism of varieties $f : X \to Y$, we will denote f^* for the derived pull-back, f_* for the derived push-forward, and so on. For a complex number $z \in \mathbb{C}$, we will write $\mathrm{Re}(z)$, resp. $\mathrm{Im}(z)$, for the real, resp. imaginary, part of z.

2. Preliminaries

In this section we realize the category $\mathbf{T}_Y$, for Y a cubic threefold, as a full subcategory of the derived category of sheaves on $\mathbb{P}^2$ with an action of an algebra $\mathcal{B}_0$. Then, by mean of this embedding, we start our study of the basic properties of $\mathbf{T}_Y$.

2.1. **Kuznetsov's theorem on quadric fibrations.** We start by recalling the notion of semi-orthogonal decomposition (see [1]). Let $\mathbf{D}$ be a triangulated category.

Definition 2.1. A *semi-orthogonal* decomposition of $\mathbf{D}$ is a sequence of full triangulated subcategories $\mathbf{T_1}, \ldots, \mathbf{T}_n \subseteq \mathbf{D}$ such that $\mathrm{Hom}_{\mathbf{D}}(\mathbf{T}_i, \mathbf{T}_j) = 0$, for $i > j$ and, for all $K \in \mathbf{D}$, there exists a chain of morphisms in $\mathbf{D}$

$$0 = K_n \to K_{n-1} \to \ldots \to K_1 \to K_0 = K$$

with $\mathrm{cone}(K_i \to K_{i-1}) \in \mathbf{T}_i$, for all $i = 1, \ldots, n$. We will denote such a decomposition by $\mathbf{D} = \langle \mathbf{T}_1, \ldots, \mathbf{T}_n \rangle$.

The easiest examples of semi-orthogonal decompositions are constructed via exceptional objects.

Definition 2.2. An object $E \in \mathbf{D}$ is called *exceptional* if $\mathrm{Hom}_{\mathbf{D}}(E, E) \cong \mathbb{C}$ and $\mathrm{Hom}^p_{\mathbf{D}}(E, E) = 0$, for all $p \neq 0$. A collection $\{E_1, \ldots, E_m\}$ of objects in $\mathbf{D}$ is called an *exceptional collection* if E_i is an exceptional object, for all i, and $\mathrm{Hom}^p_{\mathbf{D}}(E_i, E_j) = 0$, for all p and all $i > j$.

If the category $\mathbf{D}$ has good properties (for example if it is equivalent to the bounded derived category of coherent sheaves on a smooth projective variety) and $\{E_1, \ldots, E_m\}$ is an exceptional collection in $\mathbf{D}$, then we have a semi-orthogonal decomposition

$$\mathbf{D} = \langle \mathbf{T}, E_1, \ldots, E_m \rangle,$$

where, by abuse of notation, we denoted by E_i the triangulated subcategory generated by E_i (equivalent to the bounded derived category of finite dimensional vector spaces) and

$$\mathbf{T} := \langle E_1, \ldots, E_m \rangle^{\perp} = \{K \in \mathbf{T} \,:\, \mathrm{Hom}^p(E_i, K) = 0, \text{ for all } p \text{ and } i\}.$$

Let Y be a smooth hypersurface in $\mathbb{P}^4$ defined by a polynomial of degree 3 and let $\mathrm{D^b}(Y)$ be its bounded derived category of coherent sheaves. Define $\mathcal{O}_Y(1) := \mathcal{O}_{\mathbb{P}^4}(1)|_Y$ and $\mathbf{T}_Y := \langle \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle^{\perp}$. Since the collection $\{\mathcal{O}_Y, \mathcal{O}_Y(1)\}$ is exceptional in $\mathrm{D^b}(Y)$, we have a semi-orthogonal decomposition of $\mathrm{D^b}(Y)$ as

$$\mathrm{D^b}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle.$$

Remark 2.3. As Kuznetsov observed in [15], the sheaves of ideals I_l of lines l in Y are all inside the category $\mathbf{T}_Y$. Moreover, although this will not be used in this paper, all instanton bundles (and their twist by $\mathcal{O}_Y(1)$) are in $\mathbf{T}_Y$ as well.

Fix a line l_0 inside Y. Consider the following diagram:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & \widetilde{Y} := \mathrm{Bl}_{l_0} Y \arrow[r, "\pi"] \arrow[d, "\sigma"] & \mathbb{P}^2 \\ l_0 \arrow[r] & Y \subseteq \mathbb{P}^4, & \end{tikzcd}$$

where $\sigma : \widetilde{Y} \to Y$ is the blow-up of Y along l_0, D is the exceptional divisor, and $\pi : \widetilde{Y} \to \mathbb{P}^2$ is the conic fibration induced by projection from l_0 onto a plane.

Let $\mathcal{B}_0$ (resp. $\mathcal{B}_1$) be the sheaf of even (resp. odd) parts of the Clifford algebra corresponding to π, as in [16, Sect. 3]. Explicitly, in our case,

$$\mathcal{B}_0 \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-h) \oplus \mathcal{O}_{\mathbb{P}^2}(-2h)^{\oplus 2}$$
$$\mathcal{B}_1 \cong \mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-h) \oplus \mathcal{O}_{\mathbb{P}^2}(-2h),$$

as sheaves of $\mathcal{O}_{\mathbb{P}^2}$-modules, where h denotes, by abuse of notation, both the class of a line in $\mathbb{P}^2$ and its pull-back via π. Denote by $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ the abelian category of right coherent $\mathcal{B}_0$-modules and by $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$ its bounded derived category.

By [16, Sect. 4], we can define a fully faithful functor $\Phi := \Phi_{\mathcal{E}'} : \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0) \to \mathrm{D}^{\mathrm{b}}(\widetilde{Y})$, $\Phi_{\mathcal{E}'}(M) := \pi^* M \otimes_{\pi^*\mathcal{B}_0} \mathcal{E}'$, for all $M \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$, where $\mathcal{E}' \in \mathbf{Coh}(\widetilde{Y})$ is a rank 2 vector bundle on $\widetilde{Y}$ with a natural structure of flat left $\pi^*\mathcal{B}_0$-module. We will not need the actual definition of $\mathcal{E}'$ (for which the reader is referred to [16, Sect. 4]) but only the presentation

$$0 \to q^*\mathcal{B}_0(-2H) \to q^*\mathcal{B}_1(-H) \to \alpha_*\mathcal{E}' \to 0, \tag{2.1}$$

where $\alpha : \widetilde{Y} \hookrightarrow \widetilde{\mathbb{P}^4}$ is the natural embedding and $q : \widetilde{\mathbb{P}^4} \to \mathbb{P}^2$ the induced projection, $\widetilde{\mathbb{P}^4}$ being the blow-up of $\mathbb{P}^4$ along l_0 and H being, again by abuse of notation, both an hyperplane in $\mathbb{P}^4$ and its pull-backs to $\widetilde{Y}$ and to $\widetilde{\mathbb{P}}^4$. For later use, we notice that the calculation in [17, Lemma 4.1], adapted to the cubic threefolds case, yields $\mathcal{O}_{\widetilde{Y}}(D) \cong \mathcal{O}_{\widetilde{Y}}(H-h)$, $\widetilde{\mathbb{P}}^4 \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-h))$, the relative ample line bundle is $\mathcal{O}_{\widetilde{\mathbb{P}}^4}(H)$, and the relative canonical bundle is $\mathcal{O}_{\widetilde{\mathbb{P}}^4}(h-3H)$.

Recall that the left and right adjoint functors of Φ are respectively

$$\Psi(-) := \pi_*((-) \otimes \mathcal{O}_{\widetilde{Y}}(h) \otimes \mathcal{E}[1]),$$
$$\Pi(-) := \pi_*(\mathcal{H}om_{\widetilde{Y}}(\mathcal{E}', -)),$$

where $\mathcal{E} \in \mathbf{Coh}(\widetilde{Y})$ is another rank 2 vector bundle on $\widetilde{Y}$ with a natural structure of right $\pi^*\mathcal{B}_0$-module (see [16, Sect. 4]). The main property we will need is the presentation

$$0 \to q^*\mathcal{B}_1(-h-2H) \to q^*\mathcal{B}_0(-H) \to \alpha_*\mathcal{E} \to 0. \tag{2.2}$$

The embedding Φ has the remarkable property that, by [16, Thm. 4.2], it gives a semi-orthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(\widetilde{Y})$ as

$$\mathrm{D}^{\mathrm{b}}(\widetilde{Y}) = \langle \Phi(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)), \mathcal{O}_{\widetilde{Y}}(-h), \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(h) \rangle. \tag{2.3}$$

On the other hand, a well-known result of Orlov [22] tells us that

$$\mathrm{D}^{\mathrm{b}}(\widetilde{Y}) = \langle \sigma^*\mathbf{T}_Y, \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(H), \mathcal{O}_D, \mathcal{O}_D(H) \rangle, \tag{2.4}$$

where $\sigma^* : \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(\widetilde{Y})$ is fully faithful.

We now perform some mutations to compare $\mathbf{T}_Y$ and $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$, by mimicking [17, Sect. 4]. We first recall some basics about mutations (see [1]).

Let E be an exceptional object in a triangulated category $\mathbf{D}$ with good properties as before (for us, $\mathrm{D^b}(\widetilde{Y})$). Consider the two functors, respectively *left and right mutation*, $L_E, R_E : \mathbf{D} \to \mathbf{D}$ defined by

$$\begin{aligned} L_E(M) &:= \mathrm{cone}\,(ev : \mathrm{RHom}(E, M) \otimes E \to M) \\ R_E(M) &:= \mathrm{cone}\,\big(ev^\vee : M \to \mathrm{RHom}(M, E)^\vee \otimes E\big)\,[-1]. \end{aligned}$$

The main property of mutations is that, given a semi-orthogonal decomposition of $\mathbf{D}$

$$\langle \mathbf{T}_1, \ldots, \mathbf{T}_k, E, \mathbf{T}_{k+1}, \ldots, \mathbf{T}_n \rangle,$$

we can produce two new semi-orthogonal decompositions

$$\langle \mathbf{T}_1, \ldots, \mathbf{T}_k, L_E(\mathbf{T}_{k+1}), E, \mathbf{T}_{k+2}, \ldots, \mathbf{T}_n \rangle$$

and

$$\langle \mathbf{T}_1, \ldots, \mathbf{T}_{k-1}, E, R_E(\mathbf{T}_k), \mathbf{T}_{k+1}, \ldots, \mathbf{T}_n \rangle.$$

Coming back to $\mathrm{D^b}(\widetilde{Y})$, we first mutate the pair $\big(\Phi(\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)), \mathcal{O}_{\widetilde{Y}}(-h)\big)$ in (2.3), to obtain a new pair $\big(\mathcal{O}_{\widetilde{Y}}(-h), \Phi'(\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0))\big)$, with $\Phi' := R_{\mathcal{O}_{\widetilde{Y}}(-h)} \circ \Phi$. Then, by [17, Lem. 2.11], using the fact that the canonical bundle of $\widetilde{Y}$ is $\mathcal{O}_{\widetilde{Y}}(-H-h)$, we obtain a new semi-orthogonal decomposition

$$\mathrm{D^b}(\widetilde{Y}) = \langle \Phi'(\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)), \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(h), \mathcal{O}_{\widetilde{Y}}(H) \rangle. \tag{2.5}$$

On the other side, it is easy to see that the pair $\big(\mathcal{O}_{\widetilde{Y}}(H), \mathcal{O}_D\big)$ is completely orthogonal and that the left mutation of the pair $\big(\mathcal{O}_{\widetilde{Y}}(mH), \mathcal{O}_D(mH)\big)$ is $\big(\mathcal{O}_{\widetilde{Y}}(h + (m-1)H), \mathcal{O}_{\widetilde{Y}}(mH)\big)$ for all integers m. Hence, from (2.4), we obtain a new semi-orthogonal decomposition

$$\mathrm{D^b}(\widetilde{Y}) = \langle \sigma^*\mathbf{T}_Y, \mathcal{O}_{\widetilde{Y}}(h-H), \mathcal{O}_{\widetilde{Y}}, \mathcal{O}_{\widetilde{Y}}(h), \mathcal{O}_{\widetilde{Y}}(H) \rangle. \tag{2.6}$$

By comparing the two semi-orthogonal decompositions (2.5) and (2.6), we obtain a semi-orthogonal decomposition

$$\Phi'(\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)) = \langle \sigma^*\mathbf{T}_Y, \mathcal{O}_{\widetilde{Y}}(h-H) \rangle.$$

Example 2.4. As an illustration of the previous procedure, we compute the image in $\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)$ of an ideal sheaf I_l of a line l in Y which does not intersect l_0. The preliminary step is the following easy computation:

$$\Psi(\mathcal{O}_{\widetilde{Y}}(mh)) = 0, \tag{2.7}$$

for all integers m. Indeed, by the projection formula, (2.2), and the fact that $\pi = q \circ \alpha$,

$$\begin{aligned} \Psi(\mathcal{O}_{\widetilde{Y}}(mh)) &\cong \Psi(\alpha^*(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(mh))) \\ &\cong q_*(\mathcal{O}_{\widetilde{\mathbb{P}^4}}((m+1)h) \otimes \alpha_*\mathcal{E}[1]) = 0, \end{aligned}$$

for all m. Here we are using the fact that $\widetilde{\mathbb{P}^4} \to \mathbb{P}^2$ is a projective bundle and $\mathcal{O}_{\widetilde{\mathbb{P}^4}/\mathbb{P}^2}(1) \cong \mathcal{O}_{\widetilde{\mathbb{P}^4}}(H)$.

Now, to compute $(\sigma_* \circ \Phi')^{-1}(I_l)$, we first notice that, by (2.7), the mutation by $\mathcal{O}_{\widetilde{Y}}(-h)$ has no effect. More precisely, $(\sigma_* \circ \Phi')^{-1}(I_l) \cong \Psi(\sigma^* I_l)$. Now the rational map $Y \dashrightarrow \mathbb{P}^2$ is well-defined on l and maps it to another line; denote by j the embedding $l \hookrightarrow \widetilde{Y} \xrightarrow{\pi} \mathbb{P}^2$. Pulling back the exact sequence

$$0 \to I_l \to \mathcal{O}_Y \to \mathcal{O}_l \to 0,$$

we have another exact sequence

$$0 \to \sigma^* I_l \to \mathcal{O}_{\widetilde{Y}} \to \sigma^* \mathcal{O}_l = \mathcal{O}_l \to 0.$$

Again, by (2.7), we have $\Psi(\mathcal{O}_{\widetilde{Y}}) = 0$ and so

$$\begin{aligned}\Psi(\sigma^* I_l) &\cong \Psi(\mathcal{O}_l)[-1]\\ &= \pi_*(\mathcal{O}_l[-1] \otimes \mathcal{O}_{\widetilde{Y}}(h) \otimes \mathcal{E}[1]))\\ &\cong j_*(j^*\mathcal{O}_{\widetilde{Y}}(h) \otimes \mathcal{E}|_l)\\ &\cong j_*(\mathcal{E}|_l) \otimes \mathcal{O}_{\mathbb{P}^2}(h).\end{aligned}$$

2.2. **Basic properties.** In this section we collect some easy results on $\mathbf{T}_Y$ and $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$.

Let $\mathbf{D}$ be a triangulated category which arises as subcategory in a semi-orthogonal decomposition of the bounded derived category of coherent sheaves on a smooth projective variety. In particular, $\mathbf{D}$ is Ext-finite (i.e., its Hom-spaces are finite dimensional over $\mathbb{C}$), the Euler characteristic

$$\chi(-,-) := \sum_i (-1)^i \hom^i(-,-)$$

is well-defined (where $\hom^i(-,-) := \dim_{\mathbb{C}}\mathrm{Hom}(-,-[i])$), and it has a Serre functor (i.e. an autoequivalence $S : \mathbf{D} \xrightarrow{\sim} \mathbf{D}$ with functorial isomorphisms $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B, S(A))^\vee$, for all $A, B \in \mathbf{D}$).

Denote by $K(\mathbf{D})$ its Grothendieck group.

Definition 2.5. A class $[A] \in K(\mathbf{D})$ is *numerically trivial* if $\chi([M], [A]) = 0$, for all $[M] \in K(\mathbf{D})$. Define the numerical Grothendieck group $\mathcal{N}(\mathbf{D})$ as the quotient of $K(\mathbf{D})$ by numerically trivial classes.

When $\mathbf{D} = \mathrm{D}^{\mathrm{b}}(X)$, for X a smooth projective variety, we will denote $\mathcal{N}(\mathbf{D})$ by $\mathcal{N}(X)$.

Lemma 2.6. *Assume we have a semi-orthogonal decomposition $\mathbf{D} = \langle \mathbf{T}, E\rangle$, with E an exceptional object in $\mathbf{D}$. Then*

$$\mathcal{N}(\mathbf{T}) \cong \{[M] \in \mathcal{N}(\mathbf{D}) \,:\, \chi([E], [M]) = 0\}$$

and $\mathcal{N}(\mathbf{D}) \cong \mathcal{N}(\mathbf{T}) \oplus \mathbb{Z}[E]$.

Proof. The inclusion functor $\mathbf{T} \hookrightarrow \mathbf{D}$ induces a morphism $K(\mathbf{T}) \to K(\mathbf{D}) \to \mathcal{N}(\mathbf{D})$. If $[A] \in K(\mathbf{T})$ is such that $\chi(K(\mathbf{T}), [A]) = 0$, we have, for any $[K] \in K(\mathbf{D})$,

$$\chi([K], [A]) = \chi([K_T], [A]) + n\chi([E], [A]) = 0,$$

since every element $[K] \in K(\mathbf{D})$ can be written in $K(\mathbf{D})$ as $[K_T] + n[E]$, with n an integer and $[K_T] \in K(\mathbf{T})$. As a consequence we have an injective induced map $\mathcal{N}(\mathbf{T}) \hookrightarrow \mathcal{N}(\mathbf{D})$, whose image is contained in the set $[E]^\perp := \{[M] \in \mathcal{N}(\mathbf{D}) \,:\, \chi([E], [M]) = 0\}$. At the same time, if $[K] \in [E]^\perp$, then from the decomposition $[K] = [K_T] + n[E]$, it follows easily that $n = 0$, and so that $[K] \in \mathcal{N}(\mathbf{T})$. □

Proposition 2.7. *Let Y be a cubic threefold. Then*

(i) $\mathcal{N}(Y) \cong \mathbb{Z}^{\oplus 4} \cong \mathbb{Z}[\mathcal{O}_Y] \oplus \mathbb{Z}[\mathcal{O}_H] \oplus \mathbb{Z}[\mathcal{O}_l] \oplus \mathbb{Z}[\mathcal{O}_p]$, *where H is a hyperplane section, l a line, and p a point;*

(ii) $\mathcal{N}(\mathbf{T}_Y) \cong \mathbb{Z}^{\oplus 2} \cong \mathbb{Z}[I_l] \oplus \mathbb{Z}([S_{\mathbf{T}_Y}(I_l)])$, *where $S_{\mathbf{T}_Y}$ denotes the Serre functor of $\mathbf{T}_Y$. The Euler characteristic $\chi(-,-)$ on $\mathcal{N}(\mathbf{T}_Y)$ has the form, with respect to this basis,*

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Proof. For the first part, see e.g. [18, Cor. 5.8]. Let us prove the second part. By Lemma 2.6, we know that $\mathcal{N}(\mathbf{T}_Y)$ is the right orthogonal complement of the classes $[\mathcal{O}_Y]$ and $[\mathcal{O}_Y(1)]$. Hence $\mathcal{N}(\mathbf{T}_Y) \cong \mathbb{Z}^{\oplus 2}$. We have $\chi(I_l, I_l) = \chi(S_{\mathbf{T}_Y}(I_l), S_{\mathbf{T}_Y}(I_l)) = \chi(I_l, S_{\mathbf{T}_Y}(I_l)) = -1$.

By [15, Cor. 4.4], $S^3_{\mathbf{T}_Y} \cong [5]$ and so

$$\chi(S_{\mathbf{T}_Y}(I_l), I_l) = \chi(I_l, S^2_{\mathbf{T}_Y}(I_l)) = \chi(I_l, S^{-1}_{\mathbf{T}_Y}(I_l)[5]) = \chi(S_{\mathbf{T}_Y}(I_l), I_l[5]) = -\chi(S_{\mathbf{T}_Y}(I_l), I_l),$$

which means that $\chi(S_{\mathbf{T}_Y}(I_l), I_l) = 0$ and $[I_l], [S_{\mathbf{T}_Y}(I_l)]$ form a basis for $\mathcal{N}(\mathbf{T}_Y) \otimes \mathbb{Q}$. Then any $[A] \in \mathcal{N}(\mathbf{T}_Y)$ can be written as $a[I_l] + b[S_{\mathbf{T}_Y}(I_l)]$ for $a, b \in \mathbb{Q}$. But $\chi([A], [I_l]) = -a$ and $\chi([S_{\mathbf{T}_Y}(I_l)], [A]) = -b$ are integers. □

Lemma 2.8. *Let $[A]$ be a class in $\mathcal{N}(\mathbf{T}_Y)$ such that $\chi([A], [A]) = -1$. Then, up to a sign, $[A]$ is either $[I_l]$, or $[S_{\mathbf{T}_Y}(I_l)]$, or $[S^2_{\mathbf{T}_Y}(I_l)]$.*

Proof. By Proposition 2.7, a class $[A] = a[I_l] + b[S_{\mathbf{T}_Y}(I_l)]$ $(a, b \in \mathbb{Z})$ satisfies $\chi([A], [A]) = -1$ if and only if $a^2 + b^2 + ab = 1$. But this is possible if and only if either $ab = -1$ (and so $(a, b) = \pm(-1, 1)$, that means $[A] = \pm[S^2_{\mathbf{T}_Y}(I_l)]$) or $ab = 0$ (and so $(a, b) = \pm(1, 0), \pm(0, 1)$), as wanted. □

Proposition 2.9. (i) $(\Phi')^{-1}(\mathcal{O}_{\widetilde{Y}}(h - H)) \cong \mathcal{B}_1$.

(ii) *Serre duality holds for $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$ and the Serre functor $S_{\mathcal{B}_0}$ is given by $(-) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}[2]$, where $\mathcal{B}_{-1} := \mathcal{B}_1(-h)$. In particular, $S_{\mathcal{B}_0}(\mathcal{B}_1) \cong \mathcal{B}_0[2]$ and $S^2_{\mathcal{B}_0}(\mathcal{B}_1) \cong \mathcal{B}_{-1}[4]$.*

Proof. (i) As in Example 2.4, by (2.7) we have $(\Phi')^{-1}(\mathcal{O}_{\widetilde{Y}}(h - H)) \cong \Psi(\mathcal{O}_{\widetilde{Y}}(h - H))$. Then, by (2.2), we can conclude that

$$\begin{aligned}\Psi(\mathcal{O}_{\widetilde{Y}}(h - H)) &\cong \Psi(\alpha^*(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(h - H))) \\ &\cong q_*(\mathcal{O}_{\widetilde{\mathbb{P}^4}}(2h - H) \otimes \alpha_*(\mathcal{E})[1]) \\ &\cong q_*(q^*(\mathcal{B}_1)(h - 3H)[2]) \cong \mathcal{B}_1,\end{aligned}$$

where, for the last isomorphism, we used relative Serre duality with dualizing sheaf $\mathcal{O}_{\widetilde{\mathbb{P}^4}}(h - 3H)$.

(ii) The expression for the Serre functor is a standard computation using adjunction, existence of locally free resolutions, and [16, Sect. 2.1], once we observe that $\mathcal{B}_{-1} \cong \mathcal{B}_0^\vee \otimes \omega_{\mathbb{P}^2}$, where the dual is taken with respect to the $\mathcal{O}_{\mathbb{P}^2}$-module structure. For the last statement (which can also be proved by direct computation), we need to show that

$$\mathcal{B}_1 \otimes_{\mathcal{B}_0} \mathcal{B}_{-1} \cong \mathcal{B}_0,$$

which is precisely [16, Cor. 3.9]. □

Example 2.10. Consider the ideal sheaf I_{l_0} of the blown-up line l_0 in Y. We have an exact triangle in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$

$$\mathcal{B}_0[1] \to (\sigma_* \circ \Phi')^{-1}(I_{l_0}) \to \mathcal{B}_1 \xrightarrow{\eta} \mathcal{B}_0[2],$$

where η is the map corresponding to the identity of $\mathcal{B}_1$ via Serre duality.

Indeed, first of all, by [11, Prop. 11.12], we have an exact triangle

$$\mathcal{O}_D(D)[1] \to \sigma^*(\mathcal{O}_{l_0}) \to \mathcal{O}_D,$$

where as before D denotes the exceptional divisor of $\widetilde{Y}$. By (2.7), to compute $(\sigma_* \circ \Phi')^{-1}(I_{l_0})$ is sufficient to compute $\Psi(\mathcal{O}_D)$ and $\Psi(\mathcal{O}_D(D))$. To this end, we simply use (2.2) and the exact sequence

$$0 \to \mathcal{O}_{\widetilde{Y}}(h - H) \to \mathcal{O}_{\widetilde{Y}} \to \mathcal{O}_D \to 0.$$

We have

$$\begin{aligned}\Psi(\mathcal{O}_D) &\cong \Psi(\mathcal{O}_{\widetilde{Y}}(h - H))[1] \cong \mathcal{B}_1[1] \\ \Psi(\mathcal{O}_D(D)) &\cong \Psi(\mathcal{O}_{\widetilde{Y}}(H - h)) \cong \mathcal{B}_0[1].\end{aligned}$$

Hence, we have an exact triangle

$$\mathcal{B}_0[2] \to \Psi(\sigma^*\mathcal{O}_{l_0}) \to \mathcal{B}_1[1].$$

By (2.7), we have $(\sigma_* \circ \Phi')^{-1}(I_{l_0}) \cong \Psi(\sigma^* I_{l_0}) \cong \Psi(\sigma^*\mathcal{O}_{l_0})[-1]$. Hence we can write $(\sigma_* \circ \Phi')^{-1}(I_{l_0})$ as an extension of $\mathcal{B}_0[1]$ by $\mathcal{B}_1$. The fact that it is the unique non-trivial extension follows from $(\sigma_* \circ \Phi')^{-1}(I_{l_0}) \in \langle \mathcal{B}_1 \rangle^\perp$.

Proposition 2.11. *We have*
$$\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0) := \mathcal{N}(\mathrm{D^b}(\mathbb{P}^2,\mathcal{B}_0)) \cong \mathbb{Z}^{\oplus 3} \cong \mathbb{Z}[\mathcal{B}_1]\oplus\mathbb{Z}[\mathcal{B}_0]\oplus\mathbb{Z}[\mathcal{B}_{-1}].$$

Proof. By Lemma 2.6, $\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)\cong\mathcal{N}(\mathbf{T}_Y)\oplus\mathbb{Z}[\mathcal{B}_1]$. By Example 2.10, we know that $[(\sigma_*\circ\Phi')^{-1}(I_l)] = [\mathcal{B}_1]-[\mathcal{B}_0]$. At the same time, by Proposition 2.9 (ii),
$$(\sigma_*\circ\Phi')^{-1}(S_{\mathbf{T}_Y}(-)) \cong R_{\mathcal{B}_0}(S_{\mathcal{B}_0}((\sigma_*\circ\Phi')^{-1}(-))).$$
Hence
$$\begin{aligned}[(\sigma_*\circ\Phi')^{-1}(S_{\mathbf{T}_Y}([I_l]))] &= [S_{\mathcal{B}_0}((\sigma_*\circ\Phi')^{-1}(I_l))] - \chi(S_{\mathcal{B}_0}((\sigma_*\circ\Phi')^{-1}(I_l)),\mathcal{B}_0)[\mathcal{B}_0]\\ &= [\mathcal{B}_0]-[\mathcal{B}_{-1}]-(-1)[\mathcal{B}_0]\\ &= 2[\mathcal{B}_0]-[\mathcal{B}_{-1}].\end{aligned}$$
This implies, by Proposition 2.7, that $[\mathcal{B}_1]$, $[\mathcal{B}_0]$, and $[\mathcal{B}_{-1}]$ form a basis of $\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)$. □

2.3. **μ-stability.** In this section we show that there exists a notion of μ-stability on $\mathbf{Coh}(\mathbb{P}^2,\mathcal{B}_0)$ (or better for objects in $\mathbf{Coh}(\mathbb{P}^2,\mathcal{B}_0)$ which are torsion-free as sheaves on $\mathbb{P}^2$) satisfying the following properties:

(1) Harder–Narasimhan and Jordan–Hölder filtrations in μ-(semi)stable objects exist.

(2) $\mathrm{Hom}(K,\widetilde{K})=0$, if $K,\widetilde{K}$ are μ-semistable torsion-free sheaves with $\mu(K)>\mu(\widetilde{K})$.

(3) The Serre functor preserves μ-stability: If K is a torsion-free μ-semistable sheaf, then $K\otimes_{\mathcal{B}_0}\mathcal{B}_{-1}$ is μ-semistable too and
$$\mu(K\otimes_{\mathcal{B}_0}\mathcal{B}_{-1})<\mu(K).$$

(4) The exceptional object $\mathcal{B}_1$ is μ-stable.

We start by defining the numerical functions *rank* and *degree* on $\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)$. Consider the forgetful functor $\mathrm{Forg}:\mathrm{D^b}(\mathbb{P}^2,\mathcal{B}_0)\to\mathrm{D^b}(\mathbb{P}^2)$ which forgets the structure of $\mathcal{B}_0$-module. Then Forg has a left adjoint $-\otimes_{\mathcal{O}_{\mathbb{P}^2}}\mathcal{B}_0$. Hence, Forg induces a group homomorphism
$$\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)\longrightarrow\mathcal{N}(\mathbb{P}^2)\cong K(\mathbb{P}^2)$$
between the numerical Grothendieck groups of $\mathrm{D^b}(\mathbb{P}^2,\mathcal{B}_0)$ and $\mathrm{D^b}(\mathbb{P}^2)$. Define
$$\begin{aligned}\mathrm{rk}:\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)\to\mathbb{Z},&\qquad \mathrm{rk}(K):=\mathrm{rk}(\mathrm{Forg}(K))\\ \deg:\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)\to\mathbb{Z},&\qquad \deg(K):=\mathrm{c}_1(\mathrm{Forg}(K)).\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^2}(h)).\end{aligned}$$
For $K\in\mathbf{Coh}(\mathbb{P}^2,\mathcal{B}_0)$ with $\mathrm{rk}(K)\neq 0$, define the slope $\mu(K):=\deg(K)/\mathrm{rk}(K)$. Moreover, when we say that K is either torsion-free or torsion of dimension d, we always mean that $\mathrm{Forg}(K)$ has this property.

By abuse of notation, we will denote the rank and degree of an object of $\mathbf{T}_Y$ by seeing it as an object in $\mathrm{D^b}(\mathbb{P}^2,\mathcal{B}_0)$. By Lemma 2.6 they induce numerical functions $\mathcal{N}(\mathbf{T}_Y)\to\mathbb{Z}$.

Lemma 2.12. (i) *The image of the numerical function* $\mathrm{rk}:\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)\to\mathbb{Z}$ *is* $4\mathbb{Z}$.

(ii) *Let* $K\in\mathbf{Coh}(\mathbb{P}^2,\mathcal{B}_0)$. *Then* $\mathrm{rk}(K)$ *is a multiple of* 8 *if and only if* $\deg(K)$ *is even.*

(iii) *Let* $K\in\mathbf{Coh}(\mathbb{P}^2,\mathcal{B}_0)$ *be such that* $\mathrm{rk}(K)=\deg(K)=0$. *Then* $\mathrm{Hom}^i(\mathcal{B}_1,K)=0$*, for* $i\neq 0$*, and* $\mathrm{Hom}(\mathcal{B}_1,K)\neq 0$.

Proof. By Proposition 2.11, we have
$$[K]=a[\mathcal{B}_1]+b[\mathcal{B}_0]+c[\mathcal{B}_{-1}],$$
for some integer a,b,c. Then $\mathrm{rk}(K)=4(a+b+c)$ and $\deg(K)=-3a-5b-7c$. This proves (i) and (ii).

For part (iii), applying the functor $(-)\otimes_{\mathcal{B}_0}\mathcal{B}_{-1}$, we have
$$\mathrm{Hom}^i(\mathcal{B}_1,K)\cong\mathrm{Hom}^i(\mathcal{B}_0,K\otimes_{\mathcal{B}_0}\mathcal{B}_{-1})\cong\mathrm{Hom}^i(\mathcal{O}_{\mathbb{P}^2},\mathrm{Forg}(K\otimes_{\mathcal{B}_0}\mathcal{B}_{-1})).$$
Since $\mathrm{rk}(K\otimes_{\mathcal{B}_0}\mathcal{B}_{-1})=\deg(K\otimes_{\mathcal{B}_0}\mathcal{B}_{-1})=0$, the conclusion follows. □

Definition 2.13. An object $K \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ such that $\mathrm{Forg}(K)$ is torsion-free is called μ-(semi)stable if $\mu(L) < \mu(K)$ (resp. $\leq$), for all $0 \neq L \hookrightarrow K$ in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ with $\mathrm{rk}(L) < \mathrm{rk}(K)$.

By repeating literally the standard proofs (see, for example, [12, Sects. 1.2, 1.3, 1.5, 1.6]), one easily shows that the μ-stability we defined enjoys properties (1) and (2). (For a more general treatment, see [24, Sect. 3].) Moreover, for a sheaf of $\mathcal{B}_0$-modules on $\mathbb{P}^2$, the decomposition in torsion and torsion free part is compatible with the $\mathcal{B}_0$-module structure.

Lemma 2.14. *Let $K \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. Then its torsion and torsion-free part, considered as a sheaf on $\mathbb{P}^2$, have a natural structure of $\mathcal{B}_0$-module such that there is an exact sequence in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$*

$$0 \to K_{tor} \to K \to K_{tf} \to 0.$$

Proof. This can be easily proved by observing that the action $K \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{B}_0 \to K$ maps $K_{tor} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{B}_0$ into K_{tor}. □

Finally, the μ-stability we have defined enjoys properties (3) and (4).

Lemma 2.15. *Let K be a torsion-free, μ-(semi)stable sheaf in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. Then $K \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}$ is μ-(semi)stable and*

$$\mu(K \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}) = \mu(K) - \frac{1}{2}. \tag{2.8}$$

Proof. By Proposition 2.9 (ii), $\widetilde{K} := K \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}$ is a torsion free sheaf in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$.

As in the proof of Lemma 2.12, given $[K] = a[\mathcal{B}_1] + b[\mathcal{B}_0] + c[\mathcal{B}_{-1}]$, we have $\mathrm{rk}(K) = 4(a+b+c)$, $\deg(K) = -3a - 5b - 7c$, $\mathrm{rk}(\widetilde{K}) = 4(a + b + c) = \mathrm{rk}(K)$, and $\deg(\widetilde{K}) = -5a - 7b - 9c = \deg(K) - (1/2)\mathrm{rk}(K)$. From this we immediately deduce (2.8). Moreover, if $A \hookrightarrow \widetilde{K}$ is such that $\mu(A) > \mu(\widetilde{K})$ (resp. $=$), then $A \otimes_{\mathcal{B}_0} \mathcal{B}_1 \hookrightarrow K$ and $\mu(A \otimes_{\mathcal{B}_0} \mathcal{B}_1) > \mu(K)$ (resp. $=$), contradicting the μ-(semi)stability of K. Hence $\widetilde{K}$ is μ-(semi)stable, as wanted. □

Notice that, by Lemma 2.12 (i), every object in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ has rank a multiple of 4. Since $\mathcal{B}_1$ is locally-free of rank 4 it must be μ-stable, which is precisely property (4).

3. Construction of a bounded t-structure

In this section we construct a bounded t-structure on $\mathbf{T}_Y$, by proving, in various steps, the following result.

Theorem 3.1. *There exists a bounded t-structure on $\mathbf{T}_Y$ with heart $\mathbf{B}$ such that*
(i) $\mathbf{B}$ *has Ext-dimension equal to 2, i.e.* $\mathrm{Ext}^i(C, \widetilde{C}) = 0$, *for all* $C, \widetilde{C} \in \mathbf{B}$ *and for all* $i \neq 0, 1, 2$;
(ii) $I_l \in \mathbf{B}$, *for all lines* $l \subseteq Y$.

Step 1. (Bridgeland's tilting) Define a torsion pair on $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ as follows:

$$\mathcal{T}_0 := \left\{K \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0) \,:\, \text{either } K \text{ torsion or } \mu^-(K_{tf}) > \mu(\mathcal{B}_0) = -5/4\right\}$$
$$\mathcal{F}_0 := \left\{K \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0) \,:\, K \text{ torsion free and } \mu^+(K) \leq \mu(\mathcal{B}_0) = -5/4\right\},$$

where K_{tf} is defined by Lemma 2.14, and μ^+ (resp. μ^-) denotes the biggest (resp. smallest) slope of the factors in a Harder–Narasimhan filtration.

We observe here that, by (3) and (4) of Section 2.3, $\mathcal{B}_1 \in \mathcal{T}_0$ and $\mathcal{B}_0 \in \mathcal{F}_0$. By [8] we get a new bounded t-structure on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$. We denote by $\mathbf{A}_0$ its heart. Explicitly,

$$\mathbf{A}_0 = \left\{ C \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0) \,:\, \begin{array}{l} \bullet\ \mathcal{H}^i_{\mathbf{Coh}}(C) = 0, \text{ for all } i \neq 0, -1 \\ \bullet\ \mathcal{H}^0_{\mathbf{Coh}}(C) \in \mathcal{T}_0 \\ \bullet\ \mathcal{H}^{-1}_{\mathbf{Coh}}(C) \in \mathcal{F}_0 \end{array} \right\},$$

where $\mathcal{H}^\bullet_{\mathbf{Coh}}$ denotes the cohomology with respect to the t-structure with heart $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$.

Step 2. (Ext-dimension of $\mathbf{A}_0$) An important property of $\mathbf{A}_0$ is having Ext-dimension equal to 2 as $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. Indeed, for all $C, \widetilde{C} \in \mathbf{A}_0$, we have $\mathrm{Hom}^{<0}(C, \widetilde{C}) = 0$, by definition of bounded t-structure. Moreover, by construction and Proposition 2.9 (ii), $\mathrm{Hom}^{\geq 4}(C, \widetilde{C}) = 0$. Hence we only need to show that $\mathrm{Hom}^3(C, \widetilde{C}) = 0$. But an easy computation shows that

$$\mathrm{Hom}^3(C, \widetilde{C}) \cong \mathrm{Hom}^2(\mathcal{H}^{-1}_{\mathbf{Coh}}(C), \mathcal{H}^0_{\mathbf{Coh}}(\widetilde{C})) \cong \mathrm{Hom}(\mathcal{H}^0_{\mathbf{Coh}}(\widetilde{C}), \mathcal{H}^{-1}_{\mathbf{Coh}}(C) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1})^\vee.$$

By property (3) of Section 2.3, $\mathcal{H}^{-1}_{\mathbf{Coh}}(C) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1} \in \mathcal{F}_0$. Hence

$$\mathrm{Hom}(\mathcal{H}^0_{\mathbf{Coh}}(\widetilde{C}), \mathcal{H}^{-1}_{\mathbf{Coh}}(C) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}) = 0,$$

as wanted.

Step 3. ($\mathcal{B}_1$-dimension of $\mathbf{A}_0$) Another good property of $\mathbf{A}_0$ is that its $\mathcal{B}_1$-dimension is equal to 1, that is $\mathrm{Hom}^i(\mathcal{B}_1, \mathbf{A}_0) = 0$ if $i \neq 0, 1$. Indeed, by Step 2, we only need to show that $\mathrm{Hom}^2(\mathcal{B}_1, \mathbf{A}_0) = 0$. Let $C \in \mathbf{A}_0$. Then an easy computation, using Step 2, shows that

$$\mathrm{Hom}^2(\mathcal{B}_1, C) \cong \mathrm{Hom}^2(\mathcal{B}_1, \mathcal{H}^0_{\mathbf{Coh}}(C)) \cong \mathrm{Hom}(\mathcal{H}^0_{\mathbf{Coh}}(C), \mathcal{B}_0) = 0,$$

by definition of torsion pair.

Step 4. (Inducing a bounded t-structure on $\mathbf{T}_Y$)[1] Set $\mathbf{B} := (\sigma_* \circ \Phi')(\mathbf{A}_0) \cap \mathbf{T}_Y$. The following result concludes the proof of part (i) of Theorem 3.1.

Lemma 3.2. *The category* $\mathbf{B}$ *is the heart of a bounded* t*-structure on* $\mathbf{T}_Y$*.*

Proof. Consider the spectral sequence (see, for example, [21])

$$E_2^{p,q} := \bigoplus_i \mathrm{Hom}^p(\mathcal{H}^i_0(C), \mathcal{H}^{i+q}_0(\widetilde{C})) \Longrightarrow \mathrm{Hom}^{p+q}(C, \widetilde{C}), \tag{3.1}$$

where the cohomology is taken with respect to $\mathbf{A}_0$ and $C, \widetilde{C} \in \mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$. When $C = \mathcal{B}_1$, by Step 3, it degenerates at the second order. This means that, if $\widetilde{C} \in (\sigma_* \circ \Phi')^{-1}(\mathbf{T}_Y)$, then $\mathcal{H}^i_0(\widetilde{C})$ is in $(\sigma_* \circ \Phi')^{-1}(\mathbf{T}_Y)$ as well. This is enough to conclude. □

Step 5. (Ideal sheaves of lines) Here we will prove that the ideal sheaves of lines of Y belong to $\mathbf{A}_0$, thus completing the proof of Theorem 3.1.

Let $l \subseteq Y$ be a line in Y. If l does not intersect l_0, then, by Example 2.4, $(\sigma_* \circ \Phi')^{-1}(I_l) \cong j_*(\mathcal{E}|_l) \otimes \mathcal{O}_{\mathbb{P}^2}(h)$ which is a torsion sheaf supported on a line in $\mathbb{P}^2$, where j denotes the embedding given by the composition $l \hookrightarrow \widetilde{Y} \xrightarrow{\pi} \mathbb{P}^2$. Hence $(\sigma_* \circ \Phi')^{-1}(I_l)$ belongs to $\mathcal{T}_0$ and so to $\mathbf{A}_0$.

At the same time, if $l = l_0$, then, by Example 2.10, $(\sigma_* \circ \Phi')^{-1}(I_{l_0})$ is given by the unique extension $\mathcal{B}_1 \to \mathcal{B}_0[2]$. In this case, $(\sigma_* \circ \Phi')^{-1}(I_{l_0})$ is in $\mathbf{A}_0$ since $\mathcal{B}_1 \in \mathcal{T}_0$ and $\mathcal{B}_0 \in \mathcal{F}_0$.

Hence, to prove Theorem 3.1, we want to show that if $l \cap l_0 = \{pt\}$ then $(\sigma_* \circ \Phi')^{-1}(I_l)$ is again a torsion sheaf. A local computation shows that $\sigma^* I_l$ is a sheaf, sitting in an exact sequence

$$0 \to \sigma^* I_l \to \mathcal{O}_{\widetilde{Y}} \to \mathcal{O}_{l \cup \gamma} \to 0,$$

where $\gamma := \sigma^{-1}(pt)$ and l denotes, by abuse of notation, the strict transform of l inside $\widetilde{Y}$. Now, by (2.7), we have

$$(\sigma_* \circ \Phi')^{-1}(I_l) \cong \Psi(\sigma^* I_l) \cong \Psi(\mathcal{O}_{l \cup \gamma})[-1] \cong \pi_*(\mathcal{E}|_{l \cup \gamma} \otimes \mathcal{O}_{\widetilde{Y}}(h)).$$

By using the exact sequence

$$0 \to \mathcal{O}_\gamma(-h) \to \mathcal{O}_{l \cup \gamma} \to \mathcal{O}_l \to 0,$$

we have an exact triangle

$$\pi_*(\mathcal{E}|_\gamma) \to \Psi(\sigma^* I_l) \to \pi_*(\mathcal{E}|_l),$$

[1]This step, which simplifies a previous version of our argument, was suggested to us by A. Kuznetsov.

where $\pi_*(\mathcal{E}|_\gamma)$ is a torsion sheaf supported on a line in $\mathbb{P}^2$, since π is a closed embedding on γ. By (2.2), the sheaf $\mathcal{E}|_l$ has no higher cohomologies and so $\pi_*(\mathcal{E}|_l)$ is a torsion sheaf supported on a point. As a consequence, $\Psi(\sigma^* I_l)$ is a torsion sheaf on $\mathbb{P}^2$, as we wanted.

4. Objects and properties of the tilted category

The goal of this section is to prove the following result.

Theorem 4.1. *There exists a stability condition on $\mathbf{T}_Y$ such that all ideal sheaves of lines in Y are stable and they are the only stable objects in $\mathbf{B}$ in their numerical class.*

4.1. **Bridgeland's stability conditions.** We recall Bridgeland's definition of the notion of stability condition on a triangulated category. Let $\mathbf{T}$ be a triangulated category. A *stability condition* on $\mathbf{T}$ is a pair $\sigma = (Z, \mathcal{P})$ where $Z : K(\mathbf{T}) \to \mathbb{C}$ is a group homomorphism and $\mathcal{P}(\phi) \subset \mathbf{T}$ are full additive subcategories, $\phi \in \mathbb{R}$, satisfying the following conditions:

(a) If $0 \neq C \in \mathcal{P}(\phi)$, then $Z(C) = m(C)\exp(i\pi\phi)$ for some $m(C) \in \mathbb{R}_{>0}$.

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ for all $\phi \in \mathbb{R}$.

(c) If $\phi_1 > \phi_2$ and $C_i \in \mathcal{P}(\phi_i)$, $i = 1, 2$, then $\mathrm{Hom}_{\mathbf{T}}(C_1, C_2) = 0$.

(d) Any $0 \neq C \in \mathbf{T}$ admits a *Harder–Narasimhan filtration* (*HN-filtration* for short) given by a collection of distinguished triangles $C_{i-1} \to C_i \to A_i$ with $C_0 = 0$ and $C_n = C$ such that $A_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \ldots > \phi_n$.

It can be shown that each subcategory $\mathcal{P}(\phi)$ is extension-closed and abelian. Its non-zero objects are called *semistable* of phase ϕ, while the objects A_i in (d) are the *semistable factors* of C. The minimal objects of $\mathcal{P}(\phi)$ are called *stable* of phase ϕ (recall that a *minimal object* in an abelian category, also called *simple*, is a non-zero object without proper subobjects or quotients). A HN-filtration of an object C is unique up to a unique isomorphism.

For any interval $I \subseteq \mathbb{R}$, $\mathcal{P}(I)$ is defined to be the extension-closed subcategory of $\mathbf{T}$ generated by the subcategories $\mathcal{P}(\phi)$, for $\phi \in I$. Bridgeland proved that, for all $\phi \in \mathbb{R}$, $\mathcal{P}((\phi, \phi + 1])$ is the heart of a bounded t-structure on $\mathbf{T}$. The category $\mathcal{P}((0, 1])$ is called the *heart* of σ.

Remark 4.2. By [3, Prop. 5.3] giving a stability condition on a triangulated category $\mathbf{T}$ is equivalent to giving a bounded t-structure on $\mathbf{T}$ with heart $\mathbf{A}$ and a group homomorphism $Z : K(\mathbf{A}) \to \mathbb{C}$ such that $Z(C) \in \mathbb{H}$, for all $0 \neq C \in \mathbf{A}$, and with Harder–Narasimhan filtrations (see [3, Sect. 5.2]). More precisely, as $\mathbb{H} := \{z \in \mathbb{C}^* : z = |z|\exp(i\pi\phi),\ 0 < \phi \leq 1\}$, any $0 \neq C \in \mathbf{A}$ has a well-defined *phase* $\phi(C) := \arg(Z(C)) \in (0, 1]$. For $\phi \in (0, 1]$, an object $0 \neq C \in \mathbf{A}$ is then in $\mathcal{P}(\phi)$ if and only if, for all $C \twoheadrightarrow B \neq 0$ in $\mathbf{A}$, $\phi = \phi(C) \leq \phi(B)$.

A stability condition is called *locally-finite* if there exists some $\epsilon > 0$ such that, for all $\phi \in \mathbb{R}$, each (quasi-abelian) subcategory $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length. In this case $\mathcal{P}(\phi)$ has finite length so that every object in $\mathcal{P}(\phi)$ has a finite *Jordan–Hölder filtration* (*JH-filtration* for short) into stable factors of the same phase. The set of stability conditions which are locally finite will be denoted by $\mathrm{Stab}(\mathbf{T})$.

4.2. **Constructing stability conditions.** We now construct a stability condition on $\mathbf{T}_Y$, using Remark 4.2. Define a group homomorphism $Z : K(\mathbf{T}_Y) \to \mathbb{Z}$ as follows:

$$Z([C]) = \mathrm{rk}(C) + i(\deg(C) - \mu(\mathcal{B}_0)\mathrm{rk}(C)),$$

where $C \in \mathbf{B}$ is seen as an object of $\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)$, and rk and deg are the two numerical functions of Section 2.3.

Lemma 4.3. *The group homomorphism Z has the property $Z(\mathbf{B} \setminus \{0\}) \subseteq \mathbb{H}$.*

Proof. By the definition of $\mathbf{B}$ in Step 4 of the proof of Theorem 3.1, if $C \in \mathbf{B}$ is non-zero, then its image in $\mathrm{D^b}(\mathbb{P}^2, \mathcal{B}_0)$ fits into an exact triangle

$$\mathcal{H}^{-1}_{\mathbf{Coh}}(C)[1] \to C \to \mathcal{H}^0_{\mathbf{Coh}}(C),$$

where the cohomology is taken with respect to $\mathbf{Coh}(\mathbb{P}^2,\mathcal{B}_0)$. Then it will be sufficient to prove that $Z(\mathcal{H}^0_{\mathbf{Coh}}(C))$ and $Z(\mathcal{H}^{-1}_{\mathbf{Coh}}(C)[1])$ have the required property, where, by abuse of notation, Z is regarded as extended to $\mathcal{N}(\mathbb{P}^2,\mathcal{B}_0)$. But $\mathcal{H}^0_{\mathbf{Coh}}(C)$ is, by definition, in $\mathcal{T}_0$. Hence

$$\deg(\mathcal{H}^0_{\mathbf{Coh}}(C))-\mu(\mathcal{B}_0)\mathrm{rk}(\mathcal{H}^0_{\mathbf{Coh}}(C))>0.$$

Indeed, if $\mathrm{rk}(\mathcal{H}^0_{\mathbf{Coh}}(C))>0$, then $\mu(\mathcal{H}^0_{\mathbf{Coh}}(C))>\mu(\mathcal{B}_0)$. At the same time, sheaves with torsion supported on points do not belong to $\mathbf{T}_Y$, by Lemma 2.12 (iii). Hence, if $\mathrm{rk}(\mathcal{H}^0_{\mathbf{Coh}}(C))=0$, then $\deg(\mathcal{H}^0_{\mathbf{Coh}}(C))>0$.

Similarly, $\mathcal{H}^{-1}_{\mathbf{Coh}}(C)\in\mathcal{F}_0$. Hence

$$\deg(\mathcal{H}^{-1}_{\mathbf{Coh}}(C)[1])-\mu(\mathcal{B}_0)\mathrm{rk}(\mathcal{H}^{-1}_{\mathbf{Coh}}(C)[1])\geq 0.$$

If equality holds, that is $\mu(\mathcal{H}^{-1}_{\mathbf{Coh}}(C))=\mu(\mathcal{B}_0)$, then $\mathrm{rk}(\mathcal{H}^{-1}_{\mathbf{Coh}}(C)[1])<0$, as wanted. □

Lemma 4.4. *The pair $(Z,\mathbf{B})$ defines a locally finite stability condition σ on $\mathbf{T}_Y$.*

Proof. We need to show that σ has Harder–Narasimhan filtrations and is locally-finite. But the locally-finiteness condition is obvious, since the image of Z is a discrete subgroup of $\mathbb{C}$ (see [3, Sect. 5]).

The existence of Harder–Narasimhan filtrations can be proved using the same ideas as in the proof of [4, Prop. 7.1]. More precisely, the criterion we want to use is the one in [3, Prop. 2.4].

For $G\in\mathbf{B}$, denote $f(G):=\mathrm{Im}(Z(G))\geq 0$. Clearly f is additive and, if

$$0\to A\to C\to B\to 0$$

is an exact sequence in $\mathbf{B}$, then $f(A)\leq f(C)$ and $f(B)\leq f(C)$. With this in mind, let

$$\ldots\subseteq C_{j+1}\subseteq C_j\subseteq\ldots\subseteq C_1\subseteq C_0=C$$

be an infinite sequence in $\mathbf{B}$ of subobjects of C with $\phi(C_{j+1})>\phi(C_j)$, for all j. Since f is discrete, there exists $N\in\mathbb{N}$ such that $0\leq f(C_n)=f(C_{n+1})$, for all $n\geq N$. Consider the exact sequence in the category $\mathbf{B}$

$$0\to C_{n+1}\to C_n\to G_{n+1}\to 0.$$

By the additivity of f we have $f(G_{n+1})=0$, for all $n\geq N$. But this yields $\phi(G_{n+1})=1$, for all $n\geq N$ and so $\phi(C_{n+1})\leq\phi(C_n)$. This contradicts our assumptions and so, the property (a) of [3, Prop. 2.4] is verified.

Now let

$$C=C_0\twoheadrightarrow C_1\twoheadrightarrow\ldots\twoheadrightarrow C_j\twoheadrightarrow C_{j+1}\twoheadrightarrow\ldots$$

be an infinite sequence in $\mathbf{B}$ of quotients of C with $\phi(C_j)>\phi(C_{j+1})$, for all j. As before, $f(C_n)=f(C_{n+1})$, for all $n\geq N$. Consider the exact sequence in $\mathbf{B}$

$$0\to G_n\to C_N\to C_n\to 0,$$

for $n\geq N$. Then $f(G_n)=0$.

Assume we can produce a short exact sequence

$$0\to A\to C_N\to B\to 0 \tag{4.1}$$

in $\mathbf{B}$ with $A\in\mathcal{P}(1)$ and $\mathrm{Hom}(P,B)=0$, for all $P\in\mathcal{P}(1)$, where

$$\mathcal{P}(1):=\{M\in\mathbf{B}\,:\,f(M)=0\}.$$

Then $f(G_n)=0$ yields $G_n\in\mathcal{P}(1)$, for all $n\geq N$. Since $\mathrm{Hom}(G_n,B)=0$, $G_n\hookrightarrow C_N$ factorizes through $G_n\hookrightarrow A$. Hence

$$0=G_N\subseteq G_{N+1}\subseteq\ldots\subseteq G_n\subseteq\ldots\subseteq A$$

is an increasing sequence of subobjects of A in $\mathcal{P}(1)$. By the proof of Lemma 4.3, $\mathcal{P}(1)$ is of finite length: indeed its obects are shifts of μ-semistable torsion-free sheaves with slope $\mu(\mathcal{B}_0)$. Thus the

previous increasing sequence cannot exist and this proves property (b) of [3, Prop. 2.4] and so our result.

We only need to prove the existence of the exact sequence (4.1), which we do for $M \in \mathbf{B}$. Then we can write it as

$$\mathcal{H}^{-1}_{\mathbf{Coh}}(M)[1] \to M \to \mathcal{H}^{0}_{\mathbf{Coh}}(M).$$

By the description of the objects in $\mathcal{P}(1)$ mentioned above,

$$\mathrm{Hom}(\mathcal{P}(1), M) \cong \mathrm{Hom}(\mathcal{P}(1), \mathcal{H}^{-1}_{\mathbf{Coh}}(M)[1]).$$

Consider a Harder–Narasimhan filtration of $\mathcal{H}^{-1}_{\mathbf{Coh}}(M)$ in μ-semistable sheaves. If $\mu^+(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) < \mu(\mathcal{B}_0)$, then $\mathrm{Hom}(\mathcal{P}(1), M) = 0$, and we can take $A = 0$ in (4.1). If $\mu^+(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) = \mu(\mathcal{B}_0)$, then we can take for A the biggest μ-semistable subobject of $\mathcal{H}^{-1}_{\mathbf{Coh}}(M)$ in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ with slope $\mu(\mathcal{B}_0)$ and which belongs to $\mathbf{T}_Y$. Notice that its existence is ensured by the noetherianity of $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. The claim is proved. □

Corollary 4.5. *If Y is a cubic threefold, then $\mathrm{Stab}(\mathrm{D^b}(Y))$ is non-empty.*

Proof. Take the subcategory $\mathbf{D} := \langle \mathbf{T}_Y, \mathcal{O}_Y \rangle$ of $\mathrm{D^b}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1) \rangle$. By Lemma 4.4, there exists a locally finite stability condition σ on $\mathbf{T}_Y$. By construction, there exists an integer i such that $\mathrm{Hom}^{\leq i}(B, \mathcal{O}_Y) = 0$, for all $B \in \mathbf{B}$. Define trivially on the subcategory $\langle \mathcal{O}_Y[i] \rangle$ a locally finite stability condition such that $\mathcal{O}_Y[i]$ is in its heart and has phase 1. In this way, the hypotheses of [5, Prop. 3.3] are satisfied and this means that $\mathrm{Stab}(\mathbf{D}) \neq \emptyset$. Now repeat the same argument for $\mathrm{D^b}(Y) = \langle \mathbf{D}, \mathcal{O}_Y(1) \rangle$. □

4.3. **Some stable objects.** In this section we prove the stability of special objects of $\mathbf{T}_Y$.

Proposition 4.6. *Let $C \in \mathbf{T}_Y$ be such that $\mathrm{Hom}^1(C, C) \cong \mathbb{C}^2$. Then C is σ-stable.*

To prove the above proposition we first need two lemmas.

Lemma 4.7. *There exists no non-zero object $C \in \mathbf{B}$ with either $\mathrm{Hom}^1(C, C) = 0$ or $\mathrm{Hom}^1(C, C) \cong \mathbb{C}$.*

Proof. This follows easily from Proposition 2.7 (ii): indeed, $\chi(C, C) \leq -1$. Hence, since the Ext-dimension of $\mathbf{B}$ is equal to 2, if either $\mathrm{Hom}^1(C, C) = 0$ or $\mathrm{Hom}^1(C, C) \cong \mathbb{C}$, then $\chi(C, C) \geq 0$, a contradiction. □

Lemma 4.8. *Let $C \in \mathbf{T}_Y$ be such that $\mathrm{Hom}^1(C, C) \cong \mathbb{C}^2$. Then, up to shifts, C belongs to $\mathbf{B}$.*

Proof. Consider again the spectral sequence (3.1)

$$E_2^{p,q} := \bigoplus_i \mathrm{Hom}^p(\mathcal{H}^i_{\mathbf{B}}(C), \mathcal{H}^{i+q}_{\mathbf{B}}(C)) \Longrightarrow \mathrm{Hom}^{p+q}(C, C), \tag{4.2}$$

where the cohomology is taken with respect to $\mathbf{B}$. Since the Ext-dimension of $\mathbf{B}$ is equal to 2, then the $E_2^{1,q}$ terms of the spectral sequence survive. In particular, for $q = 0$ we have

$$2 = \hom^1(C, C) \geq \sum_i \hom^1(\mathcal{H}^i_{\mathbf{B}}(C), \mathcal{H}^i_{\mathbf{B}}(C)).$$

But, by Lemma 4.7,

$$\sum_i \hom^1(\mathcal{H}^i_{\mathbf{B}}(C), \mathcal{H}^i_{\mathbf{B}}(C)) \geq 2r,$$

where $r \geq 1$ is the number of non-zero cohomologies of C. Hence $r = 1$, as wanted. □

By Lemma 4.8, we can assume that C as in Proposition 4.6 is in $\mathbf{B}$. Since by Theorem 3.1 (i) the Ext-dimension of $\mathbf{B}$ is 2 and $\chi(C,C) \leq -1$, then necessarily $\chi(C,C) = -1$. By Lemma 2.8, $[C]$ is either $[I_l]$, or $[S_{\mathbf{T}_Y}(I_l)]$, or $-[S^2_{\mathbf{T}_Y}(I_l)]$. In particular, the class of $[C]$ is primitive and to prove Proposition 4.6 it suffices to show that C is σ-semistable.

If $[C] = -[S^2_{\mathbf{T}_Y}(I_l)]$, then $Z(C) = Z(I_l) - Z(S_{\mathbf{T}_Y}(I_l)) \in \mathbb{R}_{<0}$ and so it is σ-semistable. Otherwise, $\operatorname{Im}(Z(C)) = 2$ and the σ-semistability follows from the lemma below.

Lemma 4.9. *Let $C \in \mathbf{B}$ be such that $\operatorname{Hom}^1(C,C) \cong \mathbb{C}^2$ and $\operatorname{Im}(Z(C)) = 2$. Then C is σ-semistable.*

Proof. Assume C is not σ-semistable. Take $0 \to A \to C \to B \to 0$, a destabilizing sequence in $\mathbf{B}$, where B is σ-semistable. Then $\operatorname{Im}(Z(A)) + \operatorname{Im}(Z(B)) = 2$. But, by Lemma 2.12 (ii), $\operatorname{Im}(Z)$ is always even and positive. Hence $\operatorname{Im}(Z(B)) = 2$ and $\operatorname{Im}(Z(A)) = 0$, so that A is σ-semistable, and $A \cong \widetilde{A}[1]$, a shift by 1 of a μ-semistable sheaf $\widetilde{A}$ with slope equal to $\mu(\mathcal{B}_0)$. We claim that $\operatorname{Hom}^{\geq 2}(B,A) = 0$. Indeed, since A and B belong to $\mathbf{B}$, we only need to examine $\operatorname{Hom}^2(B,A)$. An easy computation shows that

$$\operatorname{Hom}^2(B,A) \cong \operatorname{Hom}^2(\mathcal{H}^{-1}_{\mathbf{Coh}}(B), \widetilde{A}) \cong \operatorname{Hom}(\widetilde{A}, \mathcal{H}^{-1}_{\mathbf{Coh}}(B) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1})^\vee,$$

where $\mathcal{H}^{\bullet}_{\mathbf{Coh}}$ denotes the cohomology sheaves of B with respect to $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. But then, by Lemma 2.15, $\mu^+(\mathcal{H}^{-1}_{\mathbf{Coh}}(B) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}) < \mu(\mathcal{B}_0) = \mu(\widetilde{A})$. Hence, $\operatorname{Hom}(\widetilde{A}, \mathcal{H}^{-1}_{\mathbf{Coh}}(B) \otimes_{\mathcal{B}_0} \mathcal{B}_{-1}) = 0$, as wanted.

Consider the abelian category $\mathbf{B}_{1/2} := \mathcal{P}((-1/2, 1/2])$ in $\mathbf{T}_Y$ and denote by $\mathcal{H}^{\bullet}_{1/2}$ the cohomology with respect to the corresponding t-structure. Then we have a triangle

$$\mathcal{H}^{-1}_{1/2}(C)[1] \to C \to \mathcal{H}^0_{1/2}(C),$$

where $\mathcal{H}^{-1}_{1/2}(C) = \widetilde{A}$ and $\mathcal{H}^0_{1/2}(C) = B$. By using once more the spectral sequence (4.2) (with respect to the heart $\mathbf{B}_{1/2}$), the vanishing of $\operatorname{Hom}^2(B,A) \cong \operatorname{Hom}^3(B,\widetilde{A})$ gives that the $E_2^{1,0}$-term survives. Hence

$$\hom^1(A,A) + \hom^1(B,B) \leq 2.$$

By Lemma 4.7 this is impossible and so C is σ-semistable. $\square$

An immediate application of Proposition 4.6 is:

Corollary 4.10. *Let $l \subseteq Y$ be any line. Then I_l is σ-stable.*

4.4. **Ideal sheaves of lines as unique stable objects in their numerical class.** To conclude the proof of Theorem 4.1 we only need to show that the ideal sheaves I_l are the only σ-stable objects in $\mathbf{B}$ in their numerical class.

Let $M \in \mathbf{B}$ be a σ-stable object with numerical class $[I_l]$ and consider it as an object in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^2, \mathcal{B}_0)$, so that, by Example 2.10, $[M] = [\mathcal{B}_1] - [\mathcal{B}_0]$.

Lemma 4.11. *We have either $M \cong (\sigma_* \circ \Phi')^{-1}(I_{l_0})$ or $M \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$.*

Proof. Assume that $\mathcal{H}^{-1}_{\mathbf{Coh}}(M) \neq 0$. Then, since $\operatorname{rk}(M) = 0$ and $\operatorname{rk}(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) \neq 0$, we have $\mathcal{H}^0_{\mathbf{Coh}}(M) \neq 0$ too and an exact triangle

$$\mathcal{H}^{-1}_{\mathbf{Coh}}(M)[1] \to M \to \mathcal{H}^0_{\mathbf{Coh}}(M). \tag{4.3}$$

We divide the proof in several steps.

Step 1. Suppose that either $\mathcal{H}^0_{\mathbf{Coh}}(M)$ or $\mathcal{H}^{-1}_{\mathbf{Coh}}(M)$ is in $\mathbf{T}_Y$. Then both the two cohomologies are in $\mathbf{T}_Y$ (and so in $\mathbf{B}$) and then (4.3) destabilizes M, because $\phi(\mathcal{H}^0_{\mathbf{Coh}}(M)) < 1/2 = \phi(M)$ (see Remark 4.2).

Step 2. Here we want to prove that $\mu^-(\mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}) \geq \mu(\mathcal{B}_1)$. By Step 1, $M_0 := \mathcal{H}^0_{\mathbf{Coh}}(M)$ does not belong to $\mathbf{T}_Y$. Hence $\operatorname{Hom}(\mathcal{B}_1, M_0) \cong \mathbb{C}^{\oplus a_0} \neq 0$ and $\operatorname{Hom}^p(\mathcal{B}_1, M_0) = 0$, for all $p \neq 0$.

Consider the evaluation map $ev_0 : \mathcal{B}_1^{\oplus a_0} \to M_0$. If it is surjective, then $\mu^-((M_0)_{tf}) \geq \mu(\mathcal{B}_1)$, as wanted.

Assume ev_0 is not surjective. Set $M_1 := \mathrm{coker}(ev_0)$. Then $\mathrm{cone}(ev_0) \in \mathbf{T}_Y$ and we have an exact triangle

$$\ker(ev_0)[1] \to \mathrm{cone}(ev_0) \to M_1.$$

As before, if $\mathrm{Hom}^p(\mathcal{B}_1, M_1) \neq 0$, then $p = 0$.

Assume $\mathrm{Hom}(\mathcal{B}_1, M_1) = 0$. Then $M_1 \in \mathbf{T}_Y$. Moreover, $M_0 \twoheadrightarrow M_1$ in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ implies $M_1 \in \mathcal{T}_0$ and so $M_1 \in \mathbf{B}$. Consider the composition

$$\psi : M \to M_0 \to M_1.$$

Then $\psi \neq 0$ and the image $\mathrm{im}(\psi)$ of ψ in the abelian category $\mathbf{B}$ is non-trivial. Since $\mathrm{im}(\psi) \hookrightarrow M_1$, where M_1 is a sheaf, $\mathrm{im}(\psi) \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. So, we found a surjection $M \twoheadrightarrow \mathrm{im}(\psi)$ in $\mathbf{B}$ with $1/2 = \phi(M) \geq \phi(\mathrm{im}(\psi))$, contradicting the σ-stability of M.

Hence, $\mathrm{Hom}(\mathcal{B}_1, M_1) \cong \mathbb{C}^{\oplus a_1} \neq 0$. Proceeding as before, we consider the evaluation map $ev_1 : \mathcal{B}_1^{\oplus a_1} \to M_1$, set $M_2 := \mathrm{coker}(ev_1)$, and so on. What we have produced is a sequence of quotients

$$M_0 \twoheadrightarrow M_1 \twoheadrightarrow M_2 \twoheadrightarrow \dots$$

in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. But now $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$ is a noetherian abelian category. Hence the sequence must terminate at M_m, for some $m \gg 0$. This means that $M_m = 0$.

Let r be the smallest integer such that $\mathrm{rk}(M_{r+1}) = 0$. Then $(M_r)_{tf}$ sits in a short exact sequence in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$

$$0 \to A \to (M_r)_{tf} \to B \to 0,$$

where $\mathcal{B}_1^{\oplus a_r} \twoheadrightarrow A$ and $\mathrm{rk}(B) = 0$. This implies $\mu^-((M_r)_{tf}) \geq \mu(\mathcal{B}_1)$. Now $(M_{r-1})_{tf}$ fits in the exact sequence

$$0 \to Q \to (M_{r-1})_{tf} \to (M_r)_{tf} \to 0,$$

where Q sits in a short exact sequence in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$

$$0 \to A' \to Q \to B' \to 0,$$

where $\mathcal{B}_1^{\oplus a_{r-1}} \twoheadrightarrow A'$ and $\mathrm{rk}(B') = 0$. Therefore, $\mu^-(Q) \geq \mu(\mathcal{B}_1)$ and then $\mu^-((M_{r-1})_{tf}) \geq \mu(\mathcal{B}_1)$. By iterating the argument, we conclude $\mu^-((M_0)_{tf}) \geq \mu(\mathcal{B}_1)$, as wanted.

Step 3. Set $r := \mathrm{rk}(\mathcal{H}^0_{\mathbf{Coh}}(M)) = \mathrm{rk}(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) \neq 0$. By Step 2, we have

$$\begin{aligned} 2 = \deg(\mathcal{B}_1) - \deg(\mathcal{B}_0) &= \deg(M) = \deg(\mathcal{H}^0_{\mathbf{Coh}}(M)) - \deg(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) \\ &= \deg(\mathcal{H}^0_{\mathbf{Coh}}(M)_{tor}) + \deg(\mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}) - \deg(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) \\ &\geq \deg(\mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}) - \deg(\mathcal{H}^{-1}_{\mathbf{Coh}}(M)) \\ &\geq \frac{r}{4}(\deg(\mathcal{B}_1) - \deg(\mathcal{B}_0)). \end{aligned}$$

But this implies that $r \leq 4$ and the inequalities are equalities. By Lemma 2.12 (i), $r = 4$. Moroever, $\mathcal{H}^0_{\mathbf{Coh}}(M)$ has torsion only on points and its torsion-free part is μ-stable of slope $\mu(\mathcal{B}_1)$, and $\mathcal{H}^{-1}_{\mathbf{Coh}}(M)$ is μ-stable too, of slope $\mu(\mathcal{B}_0)$. But, by Step 1, $\mathrm{Hom}(\mathcal{B}_1, \mathcal{H}^0_{\mathbf{Coh}}(M)) \cong \mathrm{Hom}^2(\mathcal{B}_1, \mathcal{H}^{-1}_{\mathbf{Coh}}(M)) \neq 0$.

Consider a non-zero morphism $\mathcal{B}_1 \to \mathcal{H}^0_{\mathbf{Coh}}(M)$. Then, by stability, either $\mathcal{B}_1 \to \mathcal{H}^0_{\mathbf{Coh}}(M)_{tor}$ or $\mathcal{B}_1 \hookrightarrow \mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}$. But, since $\mathcal{H}^0_{\mathbf{Coh}}(M)_{tor}$ is supported on points, if all homomorphisms $\mathcal{B}_1 \to \mathcal{H}^0_{\mathbf{Coh}}(M)$ factorize through $\mathcal{H}^0_{\mathbf{Coh}}(M)_{tor}$, then, by Lemma 2.12 (iii), $\mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}$ is in $\mathbf{T}_Y$ and so it destabilizes M. Thus $\mathcal{B}_1 \hookrightarrow \mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}$.

In the same way, by Serre duality, $\mathcal{H}^{-1}_{\mathbf{Coh}}(M) \hookrightarrow \mathcal{B}_0$. Set $T_1 := \mathcal{H}^0_{\mathbf{Coh}}(M)_{tf}/\mathcal{B}_1$ and $T_2 := \mathcal{B}_0/\mathcal{H}^{-1}_{\mathbf{Coh}}(M)$. Then we have

$$\begin{aligned}[][\mathcal{B}_1] - [\mathcal{B}_0] = [M] &= [\mathcal{H}^0_{\mathbf{Coh}}(M)] - [\mathcal{H}^{-1}_{\mathbf{Coh}}(M)] \\ &= [\mathcal{H}^0_{\mathbf{Coh}}(M)_{tor}] + [\mathcal{B}_1] + [T_1] - [\mathcal{B}_0] + [T_2],\end{aligned}$$

and so $[\mathcal{H}^0_{\mathbf{Coh}}(M)_{tor}] + [T_1] + [T_2] = 0$. But then

$$\mathcal{H}^0_{\mathbf{Coh}}(M)_{tor} = T_1 = T_2 = 0,$$

and so $\mathcal{H}^0_{\mathbf{Coh}}(M) \cong \mathcal{B}_1$ and $\mathcal{H}^{-1}_{\mathbf{Coh}}(M) \cong \mathcal{B}_0$, that is $M \cong (\sigma_* \circ \Phi')^{-1}(I_{l_0})$, as wanted. □

By the previous lemma, we can assume $M \in \mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. Consider $\Phi(M) \in \mathrm{D}^{\mathrm{b}}(\widetilde{Y})$. Since $\mathcal{E}'$ is a flat left $\pi^*\mathcal{B}_0$-module, $\Phi(M)$ is a sheaf.

Lemma 4.12. *We have*

$$\operatorname{Ext}^p(\Phi(M), \mathcal{O}_{\widetilde{Y}}(-h)) = \begin{cases} \mathbb{C}, & \text{if } p = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First of all, by adjunction,

$$\operatorname{Ext}^p(\Phi(M), \mathcal{O}_{\widetilde{Y}}(-h)) \cong \operatorname{Ext}^p(M, \Pi(\mathcal{O}_{\widetilde{Y}}(-h))).$$

We have that $\Pi(\mathcal{O}_{\widetilde{Y}}(-h)) \cong \mathcal{B}_1$. Indeed, since the normal bundle of $\widetilde{Y}$ in $\widetilde{\mathbb{P}}^4$ is $\mathcal{O}_{\widetilde{Y}}(2H+h)$, we get

$$\begin{aligned}\Pi(\mathcal{O}_{\widetilde{Y}}(-h)) &= \pi_*\mathcal{H}om(\mathcal{E}', \mathcal{O}_{\widetilde{Y}}(-h)) \\ &\cong q_*\alpha_*\mathcal{H}om(\mathcal{E}'(2H+h), \alpha^!\mathcal{O}_{\widetilde{\mathbb{P}}^4}(-h)[1]) \\ &\cong q_*\mathcal{H}om(\alpha_*(\mathcal{E}')(2H+h), \mathcal{O}_{\widetilde{\mathbb{P}}^4}(-h)[1]),\end{aligned}$$

where for the last isomorphism we use relative Serre duality. Since $\alpha_*\mathcal{E}'$ sits inside the short exact sequence (2.1), we obtain the isomorphisms

$$\Pi(\mathcal{O}_{\widetilde{Y}}(-h)) \cong \mathcal{B}_0^\vee(-2h) \cong \mathcal{B}_1.$$

Since M is a torsion sheaf, $\operatorname{Ext}^p(M, \mathcal{B}_1) = 0$, for all $p \neq 1, 2$. Moreover,

$$\chi(M, \mathcal{B}_1) = \chi(\mathcal{B}_0[1], \mathcal{B}_1) + \chi(\mathcal{B}_1, \mathcal{B}_1) = -1.$$

Hence, we only need to show that $\operatorname{Ext}^2(M, \mathcal{B}_1) = 0$.

By Proposition 2.9, $\operatorname{Ext}^2(M, \mathcal{B}_1) \cong \operatorname{Hom}(\mathcal{B}_0(h), M)^\vee$. Assume that $\operatorname{Hom}(\mathcal{B}_0(h), M) \neq 0$. We have an exact sequence in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$

$$0 \to \mathcal{B}_1 \to \mathcal{B}_0(h) \to Q \to 0, \tag{4.4}$$

where $Q := \mathcal{B}_0(h)/\mathcal{B}_1$ is pure of dimension 1. Since $M \in \mathbf{T}_Y$, the composition $\mathcal{B}_1 \to \mathcal{B}_0(h) \to M$ is the zero map. Hence we have a non-zero morphism $\eta : Q \to M$. Suppose η is not injective. Then, by Lemma 2.12 (ii), $\ker(\eta)$ has degree 2 (as $\mathrm{rk}(\ker(\eta)) = 0$ and $\deg(Q) = 2$). By Lemma 4.13, M is also pure of dimension 1 and so $\ker(\eta) = 0$. At the same time, by applying $\operatorname{Hom}(\mathcal{B}_1, -)$ to the exact sequence (4.4), we have $\operatorname{Hom}(\mathcal{B}_1, Q) \neq 0$ and then $\operatorname{Hom}(\mathcal{B}_1, M) \neq 0$, a contradiction. □

Lemma 4.13. $\operatorname{Forg}(M)$ *is a pure sheaf on* $\mathbb{P}^2$ *of dimension* 1.

Proof. Suppose the claim is not true, and denote by T the torsion part of $\operatorname{Forg}(M)$ supported in dimension 0. It is easy to see, as in Lemma 2.14, that then T has a structure of $\mathcal{B}_0$-module for which $T \hookrightarrow M$ in $\mathbf{Coh}(\mathbb{P}^2, \mathcal{B}_0)$. But then by Lemma 2.12 (iii), $\operatorname{Hom}(\mathcal{B}_1, T) \neq 0$, and so $\operatorname{Hom}(\mathcal{B}_1, M) \neq 0$, contradicting $M \in \mathbf{T}_Y$. □

Since $\Phi' = R_{\mathcal{O}_{\widetilde{Y}}(-h)} \circ \Phi$, by Lemma 4.12, $\Phi'(M)$ is given by an extension

$$0 \to \mathcal{O}_{\widetilde{Y}}(-h) \to \Phi'(M) \to \Phi(M) \to 0. \tag{4.5}$$

In particular, it is a sheaf on $\widetilde{Y}$. Moreover, $\mathrm{ch}(\Phi'(M)) = \mathrm{ch}(\sigma^*(I_l)) = (1, 0, *, *)$ and so $\mathrm{ch}(\Phi(M)) = (0, h, *, *)$.

Lemma 4.14. *$\Phi'(M)$ is torsion-free.*

Proof. Assume, for a contradiction, that $\Phi'(M)$ is not torsion-free. Then there exists an exact sequence

$$0 \to T \to \Phi'(M) \to N \to 0, \tag{4.6}$$

with T (resp. N) a torsion (resp. torsion-free) sheaf on $\widetilde{Y}$. But then $T \hookrightarrow \Phi(M)$ by (4.5). Hence, either T has dimension ≤ 1 or $\mathrm{ch}(T) = (0, h, *, *)$. The first case is untenable because, by Lemma 4.13, $\pi^*(M)$ has no torsion in dimension ≤ 1 and neither has $\Phi(M) = \pi^*(M) \otimes_{\pi^*\mathcal{B}_0} \mathcal{E}'$.

In the second case $\Phi(M)/T$ has dimension ≤ 1 and, by the Snake Lemma (applied to the diagram obtained from (4.5) and (4.6)), we get an extension

$$0 \to \mathcal{O}_{\widetilde{Y}}(-h) \to N \to \Phi(M)/T \to 0,$$

which must be trivial by Serre duality, a contradiction unless $T \cong \Phi(M)$. But this is again a contradiction since then $\Phi'(M) \cong \Phi(M) \oplus \mathcal{O}_{\widetilde{Y}}(-h)$ cannot be σ-stable. $\square$

Now, $\Phi'(M) \cong \sigma^*\Gamma$, with $\Gamma := \sigma_*(\Phi'(M)) \in \mathrm{D^b}(Y)$. We have that Γ is a torsion-free sheaf on Y. Indeed, the fact that it is a sheaf follows easily since, if not, then there exists an exact triangle

$$C_0 \to \Gamma \to C_1[-1]$$

in $\mathrm{D^b}(Y)$ with $C_0, C_1 \in \mathbf{Coh}(Y)$. But then $\mathcal{H}^0(\sigma^*(C_1)) = 0$ (where the cohomology is taken in $\mathbf{Coh}(\widetilde{Y})$), a contradiction unless $C_1 = 0$. Since σ is surjective, Γ is torsion-free.

Summing up, we have shown that $(\sigma_* \circ \Phi')(M)$ is a torsion-free sheaf on Y with Chern character equal to $\mathrm{ch}(I_l)$. But, since the Picard group of Y is isomorphic to $\mathbb{Z}$, this implies that $(\sigma_* \circ \Phi')(M)$ is the ideal sheaf of some line in Y, which completes the proof of Theorem 4.1.

5. A categorical Torelli theorem

In this section we prove the main result of this paper. In order to do that, we show that the ideal sheaves of lines on a cubic threefold are preserved by the action of an equivalence (up to composing with a suitable power of the Serre functor, followed by a shift). We complete the proof of Theorem 1.1 in Section 5.2, by showing that any such equivalence induces an isomorphism between the Fano surfaces of lines.

The section ends with a discussion about a generalization of Theorem 1.1 to cubic fourfolds containing a plane.

5.1. **Ideals of lines and equivalences.** Let Y and Y' be two smooth and projective cubic threefolds and let $U : \mathbf{T}_{Y'} \xrightarrow{\sim} \mathbf{T}_Y$ be an exact equivalence. Take $I_{l'}$ an ideal sheaf of a line l' in Y' and consider it as an object in $\mathbf{T}_{Y'}$. Then consider the object $U(I_{l'}) \in \mathbf{T}_Y$. Since U is an equivalence of categories, the numerical class c of $U(I_{l'})$ satisfies $\chi(c, c) = -1$. By Lemma 2.8, up to composing with some power of the Serre functor of $\mathbf{T}_Y$, we can assume $c = [I_l]$, where l is a line in Y. Moreover, by Lemma 4.8, up to shift, we can assume $U(I_{l'}) \in \mathbf{B}$, where $\mathbf{B}$ is the heart constructed in Theorem 3.1. Let $\sigma = (Z, \mathcal{P})$ be the stability condition constructed in Section 4. Then, by Proposition 4.6, $U(I_{l'})$ is σ-stable.

Now, given two lines l', m' in Y', assume $U(I_{l'})$ is σ-stable with phase $1/2$ and $U(I_{m'})$ is σ-stable of phase $\phi \in \mathbb{R}$. Then $\mathrm{Hom}(I_{m'}, I_{l'}[1]), \mathrm{Hom}(I_{l'}, I_{m'}[1]) \neq 0$ and property (c) in the definition of a stability condition give the bound $-1/2 < \phi < 3/2$. But then $\phi = 1/2$ and so, by Theorem 4.1,

there exist two lines l, m in Y such that $U(I_{l'}) \cong I_l$ and $U(I_{m'}) \cong I_m$ in $\mathbf{T}_Y$. In summary, we proved the following result.

Proposition 5.1. *Let Y and Y' be two cubic threefolds such that $\mathbf{T}_{Y'} \cong \mathbf{T}_Y$. Then there exists an equivalence $U : \mathbf{T}_{Y'} \xrightarrow{\sim} \mathbf{T}_Y$ which maps ideal sheaves of lines in Y' bijectively onto ideal sheaves of lines in Y.*

5.2. **Proof of Theorem 1.1.** We can now prove our main theorem. Indeed the moduli space of ideal sheaves of lines in a cubic threefold Y is isomorphic to the Fano surface of lines $F(Y)$, which characterizes the isomorphism class of a cubic threefold, via the classical Torelli Theorem [7]. More precisely, from $F(Y)$ one can recover the intermediate Jacobian $J(Y)$ as the Albanese variety and the natural polarization on $J(Y)$ as the class of the image of $F(Y) \times F(Y)$ in $J(Y)$ via the map $(s,t) \mapsto s - t$ (see [7]).

To prove Theorem 1.1, we only need to show that the bijection induced in Proposition 5.1 is a morphism of algebraic varieties. For a cubic threefold Y, following [19, 20], define a functor

$$Fano_Y : (\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Set})$$

by sending a $\mathbb{C}$-scheme S to the set of equivalence classes of relatively perfect complexes $\mathcal{I} \in \mathrm{D^b}(Y \times S)$ (cf. [20, Def. 2.1.1 and Cor. 4.3.4]) such that, for all $s \in S$, $\mathcal{I}|_{Y \times s}$ is an ideal sheaf of a line in Y. It is a standard fact that $Fano_Y$ is represented by $F(Y)$, which is a smooth projective surface.

Consider the universal family $\mathcal{I}' \in \mathbf{Coh}(Y' \times F(Y'))$ and take the composite functor

$$R : \mathrm{D^b}(Y') \xrightarrow{\rho'} \mathbf{T}_{Y'} \xrightarrow{U} \mathbf{T}_Y \xrightarrow{\epsilon} \mathrm{D^b}(Y),$$

where ϵ is the embedding of $\mathbf{T}_Y$ into $\mathrm{D^b}(Y)$, ρ' is the natural projection from $\mathrm{D^b}(Y')$ to $\mathbf{T}_{Y'}$ and $U : \mathbf{T}_{Y'} \xrightarrow{\sim} \mathbf{T}_Y$ is an exact equivalence as in the previous section.

As a first step, let $L \in \mathrm{Pic}(F(Y'))$ be an ample line bundle and consider the (infinite) resolution of $\mathcal{I}'$

$$\cdots \to \mathcal{O}_{Y'}(-N_i)^{\oplus n_i} \boxtimes (L^{-r_i})^{\oplus s_i} \to \cdots \to \mathcal{O}_{Y'}(-N_0)^{\oplus n_0} \boxtimes (L^{-r_0})^{\oplus s_0} \to \mathcal{I}' \to 0, \tag{5.1}$$

with the assumptions $r_i \in \mathbb{N}$ and $N_i \gg 0$ so that, if $p \neq 3$,

$$\mathrm{Hom}(\mathcal{O}_{Y'}, \mathcal{O}_{Y'}(-N_i)[p]) = \mathrm{Hom}(\mathcal{O}_{Y'}(1), \mathcal{O}_{Y'}(-N_i)[p]) = 0.$$

Choose m sufficiently large and truncate (5.1), getting a bounded complex

$$O_m^\bullet := \{\mathcal{O}_{Y'}(-N_m)^{\oplus n_m} \boxtimes (L^{-r_m})^{\oplus s_m} \to \cdots \to \mathcal{O}_{Y'}(-N_0)^{\oplus n_0} \boxtimes (L^{-r_0})^{\oplus s_0}\}. \tag{5.2}$$

Then the exact triangle

$$K_m[m] \to O_m^\bullet \to \mathcal{I}',$$

for $K_m := \ker(O_m^m \to O_m^{m-1}) \in \mathbf{Coh}(Y' \times F(Y'))$, splits. Hence $O_m^\bullet$ has a (unique) convolution $\mathcal{I}' \oplus K_m[m]$. (For general facts about convolutions, see, for example, [6, 14, 23].)

Consider the complex

$$R_m^\bullet := \{R(\mathcal{O}_{Y'}(-N_m))^{\oplus n_m} \boxtimes (L^{-r_m})^{\oplus s_m} \to \cdots \to R(\mathcal{O}_{Y'}(-N_0))^{\oplus n_0} \boxtimes (L^{-r_0})^{\oplus s_0}\}.$$

of objects in $\mathrm{D^b}(Y \times F(Y'))$.

Lemma 5.2. *The complex $R_m^\bullet$ admits a unique (up to isomorphism) split right convolution $\mathcal{G}_m = \mathcal{E}_m \oplus \mathcal{F}_m$ such that, for some $M < m$, $\mathcal{H}^i(\mathcal{E}_m) = 0$ unless $i \in [-M, 0]$ and $\mathcal{H}^i(\mathcal{F}_m) = 0$ unless $i \in [-m-M, -m]$.*

Proof. Due to [14, Lemmas 2.1, 2.4], $R_m^\bullet$ has a right convolution if

$$\mathrm{Hom}(R(\mathcal{O}_{Y'}(-N_a))^{\oplus n_a} \boxtimes (L^{-r_a})^{\oplus s_a}, R(\mathcal{O}_{Y'}(-N_b))^{\oplus n_b} \boxtimes (L^{-r_b})^{\oplus s_b}[p]) = 0$$

for $a > b$ and $p < 0$.

To show this, it is enough to prove, using Künneth decomposition, that, for $N, P \geq 3$ and $p < 0$,

$$\mathrm{Hom}(T_N, T_P[p]) = 0,$$

where $T_i := (\epsilon \circ \rho)(\mathcal{O}_{Y'}(-i))$, $i = N, P$. For this, consider the left mutation of $\mathcal{O}_{Y'}(-i)$ with respect to $\mathcal{O}_{Y'}(1)$

$$\mathcal{O}_{Y'}(1)^{\oplus s_i}[-3] \to \mathcal{O}_{Y'}(-i) \to \widetilde{T}_i, \tag{5.3}$$

where $s_i := \hom^3(\mathcal{O}_{Y'}(1), \mathcal{O}_{Y'}(-i))$. Applying to (5.3) the functor $\mathrm{Hom}(\mathcal{O}_{Y'}, -)$, we have that $\mathrm{Hom}^p(\mathcal{O}_{Y'}, \widetilde{T}_i) = 0$, unless $p = 2, 3$. Hence, by mutating with respect to $\mathcal{O}_{Y'}$, we get another triangle

$$\mathcal{O}_{Y'}^{\oplus t_i^3}[-3] \oplus \mathcal{O}_{Y'}^{\oplus t_i^2}[-2] \to \widetilde{T}_i \to T_i,$$

where $t_i^j := \hom^j(\mathcal{O}_{Y'}, \widetilde{T}_i)$.

Assume that there is a non-zero map $\phi : T_N \to T_P[-k]$, for $k > 0$. Consider the following diagram

$$\begin{tikzcd}
\widetilde{T}_N \arrow[r, "\alpha"] & T_N \arrow[r, "\beta"] \arrow[d, "\phi"] & \mathcal{O}_{Y'}^{\oplus t_N^3}[-2] \oplus \mathcal{O}_{Y'}^{\oplus t_N^2}[-1] \\
\widetilde{T}_P[-k] \arrow[r] & T_P[-k] \arrow[r, "\gamma"] & \mathcal{O}_{Y'}^{\oplus t_P^3}[-2-k] \oplus \mathcal{O}_{Y'}^{\oplus t_P^2}[-1-k].
\end{tikzcd}$$

By (5.3), it is an easy check that $\gamma \circ \phi \circ \alpha = 0$. Hence, we can lift ϕ to a morphism $\widetilde{\phi} : \widetilde{T}_N \to \widetilde{T}_P[-k]$.

If we show that $\widetilde{\phi} = 0$, then ϕ factors through β, and so it is zero by orthogonality. But, looking at the diagram

$$\begin{tikzcd}
\mathcal{O}_{Y'}(-N) \arrow[r] & \widetilde{T}_N \arrow[r] \arrow[d, "\widetilde{\phi}"] & \mathcal{O}_{Y'}(1)^{\oplus s_N}[-2] \\
\mathcal{O}_{Y'}(-P)[-k] \arrow[r] & \widetilde{T}_P[-k] \arrow[r] & \mathcal{O}_{Y'}(1)^{\oplus s_P}[-2-k],
\end{tikzcd}$$

the same argument shows that $\widetilde{\phi}$ is indeed zero, as wanted.

The splitting of the convolution follows from a standard argument (see, e.g., [6, Sect. 4.2]). $\square$

Set $\mathcal{I} := \mathcal{E}_m$. Let s be a closed point in $F(Y')$ and denote by $i_s : Y \times \{s\} \hookrightarrow Y \times F(Y')$ and $i'_s : Y' \times \{s\} \hookrightarrow Y' \times F(Y')$ the natural inclusions. We show $i_s^*(\mathcal{I}) \cong R((i'_s)^*\mathcal{I}')$.

Applying the functor i_s^* to $R_m^\bullet$, we get the complex

$$i_s^*(R_m^\bullet) := \{R(\mathcal{O}_{Y'}(-N_m))^{\oplus n_m} \otimes \mathbb{C}^{\oplus s_m} \to \cdots \to R(\mathcal{O}_{Y'}(-N_0))^{\oplus n_0} \otimes \mathbb{C}^{\oplus s_0}\}. \tag{5.4}$$

of objects in $\mathrm{D^b}(Y)$. It is easy to see that the objects $i_s^*\mathcal{I} \oplus i_s^*\mathcal{F}_m$ and $R((i'_s)^*\mathcal{I}') \oplus R((i'_s)^*K_m)[m]$ are both right convolutions of $i_s^*(R_m^\bullet)$. On the other hand, the same argument as in the proof of Lemma 5.2, shows that $i_s^*(R_m^\bullet)$ has a unique (up to isomorphism) right convolution and hence, by the choice of $m \gg 0$, $i_s^*(\mathcal{I}) \cong R((i'_s)^*\mathcal{I}')$, for any closed point $s \in F(Y')$.

This yields a morphism $F(Y') \to F(Y)$ which, being induced by R, is actually a bijection. Hence $F(Y') \cong F(Y)$ and the proof of Theorem 1.1 is complete.

5.3. **A higher dimensional example: cubic fourfolds containing a plane.** Let us consider now the case of cubic fourfolds. The richer geometry of these varieties allows one to give a geometric incarnation of the triangulated category $\mathbf{T}_Y$ sitting inside the derived category $\mathrm{D^b}(Y)$.

To be precise, let $Y \subseteq \mathbb{P}^5$ be a smooth cubic fourfold containing a plane P. The projection from P yields a rational map $\pi : Y \dashrightarrow \mathbb{P}^2$ and the blow-up of P gives a quadric fibration $\pi' : \mathcal{Q} \to \mathbb{P}^2$ whose fibres are singular along a sextic $C \subseteq \mathbb{P}^2$. The double cover S of $\mathbb{P}^2$ ramified along such a curve is a K3 surface. The cubic fourfolds containing a plane that we will study according to [17] are those satisfying the following additional hypothesis:

($*$) C is smooth, so that S is smooth as well and the fibres of π' (and of π) have at most one singular point.

One of the key geometric properties of these varieties is:

Proposition 5.3. **([25], Sect. 1, Proposition 4).** *The cubic fourfold Y is determined by the sextic C and an odd theta-characteristic θ, i.e., a line bundle $\theta \in \mathrm{Pic}(C)$ such that $\theta^{\otimes 2} \cong \omega_C$ and $h^0(C,\theta)$ is odd.*

Remark 5.4. The condition ($*$) can be relaxed allowing singular curves as well. In this case, one has to study carefully the configuration of the plane sextic and of the odd theta-characteristic in Proposition 5.3. All the results mentioned in the rest of this section continue to hold true in this more general setting and are proved in the same way.

By [17, Thm. 4.3], there exists a semi-orthogonal decomposition

$$\mathrm{D^b}(Y) = \langle \mathbf{T}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2)\rangle, \tag{5.5}$$

and an equivalence $\mathbf{T}_Y \cong \mathrm{D^b}(S,\alpha)$, where $\alpha \in \mathrm{Br}(S)$ is an element in the Brauer group of the K3 surface S. The geometric meaning of α is the following. The quadric fibration $\mathcal{Q}$ gives a $\mathbb{P}^1$-bundle D over S parametrizing lines contained in the fibres of π'. The fibration D is a Brauer–Severi variety and is hence determined by the choice of an element in $\mathrm{Br}(S)$. Since the fibres are projective lines, the order of α is 2.

Proposition 5.5. *There exist rational cubic fourfolds Y_1, Y_2 such that $\mathbf{T}_{Y_1}$ is not equivalent to $\mathbf{T}_{Y_2}$.*

Proof. By [10], there exists a codimension one subvariety in the moduli space of cubic fourfolds containing a plane consisting of rational cubic fourfolds. Moreover, in this subvariety there are all cubic fourfolds Y containing a plane such that in the Néron-Severi group $\mathrm{NS}_2(Y) := H^4(Y,\mathbb{Z}) \cap H^{2,2}(Y)$ there is a class T with the property that the intersection $T \cdot Q$ is odd, where Q is the class of a quadric in the fibre of $\pi : Y \dashrightarrow \mathbb{P}^2$.

Take two such cubic fourfolds Y_1 and Y_2 with the additional requirement that the lattices L_1 and L_2 which are the saturations of $\langle H_1^2, Q_1, T_1\rangle$ and $\langle H_2^2, Q_2, T_2\rangle$ have different discriminant greater than 8 (here H_1 and H_2 are the hyperplane sections of Y_1 and Y_2) and coincide with $\mathrm{NS}_2(Y_1)$ and $\mathrm{NS}_2(Y_2)$. Recall that the discriminant of L_i is just the order of the finite group $L_i^\vee/L_i$.

Let us show that Y_1 and Y_2 satisfy ($*$) (or better the relaxed condition in Remark 5.4), i.e. the singular locus of the fibres of $\pi' : \mathcal{Q}_i \to \mathbb{P}^2$ is at most one point. Suppose, without loss of generality, that $\mathcal{Q}_1$ contains a fiber Q which is union of two (distinct) planes P_1 and P_2. An easy computation shows that

$$P_i \cdot P = \frac{1}{2}(Q\cdot P) = -1 \qquad \text{and} \qquad P_1 \cdot P_2 = \frac{1}{2}(Q^2 - P_1^2 - P_2^2) = -1,$$

for $i \in \{1,2\}$. In particular, P, P_1 and P_2 are distinct classes in $\mathrm{NS}_2(Y_1)$ and the sublattice N of $\mathrm{NS}_2(Y_1)$ which is the saturation of the lattice $\langle H^2, P, P_1, P_2\rangle$ has rank bigger than 3, contradicting the choice of Y_1. The case where a fibre degenerates to a double plane is similar and left to the reader.

Since Y_i is rational, by [17, Prop. 4.7], $\mathbf{T}_{Y_i}$ is equivalent to $\mathrm{D^b}(S_i)$ ($i = 1,2$), and hence we need to prove that $\mathrm{D^b}(S_1) \not\cong \mathrm{D^b}(S_2)$. A result of Orlov ([23]) shows that this happens if and only if the transcendental lattices $T(S_1) := \mathrm{Pic}(S_1)^\perp$ and $T(S_2) := \mathrm{Pic}(S_2)^\perp$ are not Hodge isometric. By the results in [25, Sect. 1], the lattice $T(S_i)$ has the same discriminant as L_i. Since these discriminants are different, $T(S_1) \not\cong T(S_2)$. □

Recall that a cubic fourfold Y containing a plane P is *generic* if the group of codimension-2 algebraic classes $\mathrm{NS}_2(Y) \subseteq H^4(Y,\mathbb{Z})$ is generated by the class of P and by H^2, where $H \in H^2(Y,\mathbb{Z})$ is the class of a hyperplane section of Y. By the calculations in the proof of Proposition 5.5, these

fourfolds satisfy condition $(*)$. Kuznetsov's Conjecture 1.2 predicts that a generic cubic fourfold Y containing a plane P is not rational since $\mathrm{D}^{\mathrm{b}}(S,\alpha)$ is not equivalent to the derived category of any un-twisted K3 surface (see [17, Prop. 4.8]).

The following result gives an analogue for cubic fourfolds containing a plane of the Torelli theorem for cubic threefolds proved in this paper.

Proposition 5.6. *Given a cubic fourfold Y containing a plane P and satisfying $(*)$, there exist only finitely many isomorphism classes of cubic fourfolds $Y_1 = Y, Y_2, \ldots, Y_n$ containing a plane and with the property $(*)$ such that $\mathbf{T}_Y \cong \mathbf{T}_{Y_j}$, with $j \in \{1, \ldots, n\}$. Moreover, if Y is generic, then $n = 1$.*

Proof. Let Y' be a cubic fourfold such that $\mathbf{T}_{Y'} \cong \mathbf{T}_Y$ and containing a plane P' giving an equivalence $\mathbf{T}_{Y'} \cong \mathrm{D}^{\mathrm{b}}(S',\alpha')$ and hence an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(S,\alpha) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S',\alpha')$. By [13, Thm. 0.4], Φ induces a Hodge isometry $\Phi^* : T(S,\alpha) \xrightarrow{\sim} T(S',\alpha')$, where $T(S,\alpha)$ and $T(S',\alpha')$ are the generalized transcendental lattices. To be precise, these lattices depend on the choice of B-field lifts B and B' of the Brauer classes α and α'.

By [25, Sect. 1] (see, in particular, [25, Sect. 1, Prop. 3]), the weight-2 Hodge structure on $T(S,\alpha)$ determines the Hodge structure on $H^4(Y,\mathbb{Z})$, since $T(S,\alpha)(-1)$ is realized as a primitive sublattice of the orthogonal complement $L := \langle H^2, P\rangle^\perp$ in $H^4(Y,\mathbb{Z})$. Again H and P are respectively the hyperplane section of Y and the plane contained in Y. (Recall that, given a lattice L with quadratic form b_L, the lattice $L(-1)$ coincides with L as a group but its quadratic form $b_{L(-1)}$ is such that $b_{L(-1)} = -b_L$.)

This means that, by the Torelli theorem for cubic fourfolds (see [25]), we are reduced to proving that there are only finitely many primitive sublattices T of the sublattice $L := \langle H^2, P\rangle^\perp$ in $H^4(Y,\mathbb{Z})$ with any isometry $\varphi : T(S,\alpha)(-1) \xrightarrow{\sim} T$ which does not extend to an isometry $\overline{\varphi} : L \xrightarrow{\sim} L$, fixing the class H^2. But this is a standard result which can be found, for example, in the proof of [13, Cor. 4.6].

For the second part of the statement, assume there exists an equivalence $\Phi : \mathrm{D}^{\mathrm{b}}(S_1,\alpha_1) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(S_2,\alpha_2)$ inducing, as before, a Hodge isometry $\Phi^* : T(S_1,\alpha_1) \xrightarrow{\sim} T(S_2,\alpha_2)$. Take a B-field lift B_i for α_i. (See [13] for more details.)

We want to show that there is a Hodge isometry $f : T(S_1) \to T(S_2)$ making the following diagram commutative:

(5.6)
$$\begin{tikzcd}
0 \arrow[r] & T(S_1,\alpha_1) \arrow[r, "i_1"] \arrow[d, "\Phi^*"'] & T(S_1) \arrow[r, "\wedge B_1"] \arrow[d, dotted, "f"'] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T(S_2,\alpha_2) \arrow[r, "i_2"] & T(S_2) \arrow[r, "\wedge B_2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

First of all, observe that, up to considering the composition $i_1' := i_2 \circ \Phi^* : T(S_1,\alpha_1) \hookrightarrow T(S_2)$, we can assume $T(S_2,\alpha_2) = T(S_1,\alpha_1)$. Thus, let $\tau \in T(S_1,\alpha_1) \otimes \mathbb{Q}$ be such that $i_1(\tau)$ generates $T(S_1)$ modulo $T(S_1,\alpha_1)$. Obviously, $\tau' := 2\tau \in i_1(T(S_1,\alpha_1))$. Define f by

$$f(i_1(\tau)) := \frac{1}{2}(i_1'(i_1^{-1}(2\tau))) \qquad \text{and} \qquad f(t) := i_2(\Phi^*(i_1^{-1}(t))),$$

for any $t \in i_1(T(S_1,\alpha_1))$.

The morphism f is obviously an isometry, since the $\mathbb{Q}$-linear extension of $i_1' \circ i_1^{-1}$ is. For the same reason, f preserves the weight-2 Hodge structure on $T(S_1)$ and $T(S_2)$.

Since S_1 and S_2 are generic K3 surfaces with $\mathrm{Pic}(S_i) \cong \mathbb{Z}$ generated by an element with self-intersection 2, the isometry f extends to a Hodge isometry

$$f' : H^2(S_1,\mathbb{Z}) \xrightarrow{\sim} H^2(S_2,\mathbb{Z}).$$

Up to composing f' with $-\mathrm{id}$ and changing B_2 with $-B_2$, there exists an isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ such that $f' = \varphi_*$. (Notice that changing B_2 with $-B_2$ is no problem since $\exp(B_2) = \exp(-B_2) = \alpha_2$ as α_2 has order 2.) In other words, by (5.6), $\varphi^*(\alpha_2) = \alpha_1$.

For $i \in \{1, 2\}$, define, as before, $L_i := \langle H_i^2, P_i\rangle^{\perp_{H^4(Y_i,\mathbb{Z})}}$, where H_i and P_i are, respectively, the classes of an hyperplane section of Y_i and of the plane contained in Y_i. Again, by the discussion in [25, Sect. 1] (see, in particular, [25, Sect. 1, Prop. 3]), there exists a short exact sequence

$$0 \longrightarrow L_i(-1) \longrightarrow T(S_i) \xrightarrow{\wedge B_i} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0. \tag{5.7}$$

In [25, Sect. 2], it is shown that, given Y_i and S_i, there exists a natural isomorphism between the affine space (over $\mathbb{Z}/2\mathbb{Z}$) of the theta-characteristics on the sextic C_i along which the double cover S_i of $\mathbb{P}^2$ ramifies and the extension classes as in (5.7). In particular, the isomorphism $\varphi : S_1 \xrightarrow{\sim} S_2$ leads to isomorphic sextics and theta-characteristics. Applying Proposition 5.3, we get the desired isomorphism $Y_1 \cong Y_2$. □

Remark 5.7. For non-generic cubic fourfolds containing a plane one cannot expect that the derived category $\mathrm{D^b}(S, \alpha)$ determines the fourfold Y up to isomorphism. Indeed, using the properties of the moduli space of cubic fourfolds in [9], it is possible to construct examples of fourfolds Y_1 and Y_2 with a Hodge isometry $T(S_1, \alpha_1) \cong T(S_2, \alpha_2)$ but such that $L_1 := \langle H_1^2, P_1\rangle^\perp$ and $L_2 := \langle H_2^2, P_2\rangle^\perp$ are not isometric.

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